

LACUNARITY OF SELF-SIMILAR AND STOCHASTICALLY SELF-SIMILAR SETS

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ABSTRACT. Let K be a self-similar set in $\mathbb{R}^d$, of Hausdorff dimension D , and denote by $|K(\epsilon)|$ the d -dimensional Lebesgue measure of its ϵ -neighborhood. We study the limiting behavior of the quantity $\epsilon^{-(d-D)}|K(\epsilon)|$ as $\epsilon \rightarrow 0$. It turns out that this quantity does not have a limit in many interesting cases, including the usual ternary Cantor set and the Sierpinski carpet. We also study the above asymptotics for stochastically self-similar sets. The latter results then apply to zero-sets of stable bridges, which are stochastically self-similar (in the sense of the present paper), and then, more generally, to level-sets of stable processes. Specifically, it follows that, if K_t is the zero-set of a real-valued stable process of index $\alpha \in (1, 2]$, run up to time t , then $\epsilon^{-1/\alpha}|K_t(\epsilon)|$ converges to a constant multiple of the local time at 0, simultaneously for all $t \geq 0$, on a set of probability one.

The asymptotics for deterministic sets are obtained via the renewal theorem. The renewal theorem also yields asymptotics for the mean $\mathbb{E}[|K(\epsilon)|]$ in the random case, while the almost sure asymptotics in this case are obtained via an analogue of the renewal theorem for branching random walks.

1. INTRODUCTION

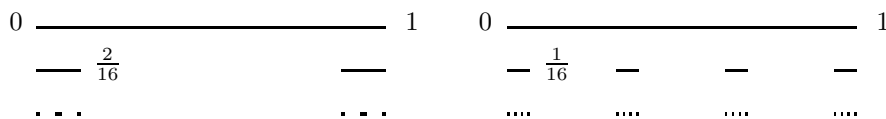
Consider the following construction of a Cantor set on the real line. Start with the unit interval $I = [0, 1]$ and divide it into $n \geq 3$ equal subintervals, each of length $r = n^{-1}$. Let N be an integer with $1 < N < n$ and keep any N of the n subintervals $I_i = [in^{-1}, (i+1)n^{-1}]$, say $I_{i_1}, \dots, I_{i_N}$, and discard the remaining ones. Next divide each of the intervals I_{i_k} kept in the first step into n equal sub-subintervals, of length $r^2 = n^{-2}$ now, and keep only N of these within each I_{i_k} , always according to the same pattern as in the first step of the construction. Continuing ad infinitum, this construction leads to a Cantor set K on the line, of Hausdorff and Minkowski (or box) dimension equal to $\log_n N$. Alternatively, $K = \{ \sum_{j=1}^{\infty} x_j n^{-j} : x_j \in \{i_1, \dots, i_N\} \}$.

Now consider two particular cases, both with $r = 1/16$ and $N = 4$, but different patterns. For the first construction keep the two leftmost and two rightmost of the subintervals in each step. For the second construction use the pattern in which the four subintervals kept are uniformly spread out; i.e., we keep the 1st, 6th, 11th and 16th. In both constructions the limit set has dimension $D = \frac{1}{2}$ but according

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FIGURE 1. Two Cantor sets with $D = \frac{1}{2}$.

to Mandelbrot ([19], page 313) these two limit sets look quite different (see Figure 1): the outcome of the first construction looks like a ‘few points’, hence ‘mimics the dimension $D = 0$ ’, while the outcome of the second construction looks like a ‘full interval’ and hence ‘mimics the dimension $D = 1$ ’. He calls the set of the first construction a set of ‘high lacunarity’ and the set of the second construction a set of ‘low lacunarity’.

In an attempt to give a numerical measure for lacunarity Mandelbrot considers the length of the ϵ -neighborhood, $|K(\epsilon)|$, of the limit set K . It is well known that $|K(\epsilon)| \approx \epsilon^{1-D}$, in the sense that

$$\lim_{\epsilon \rightarrow 0} \frac{\log |K(\epsilon)|}{\log \epsilon} = 1 - D,$$

giving the Minkowski or box dimension D of K . Mandelbrot observes that in ‘many’ cases one has the stronger relation $|K(\epsilon)| \sim L\epsilon^{1-D}$, in the sense that the ratio of the two sides tends to 1 as $\epsilon \rightarrow 0$; he calls L the ‘prefactor’ and proposes L^{-1} as a numerical measure of lacunarity, provided of course it exists.

To put it in a different way: the asymptotic ratio of the logarithms of $|K(\epsilon)|$ and ϵ gives some crude information about K , relating only to its dimension; by considering the more delicate direct ratio $|K(\epsilon)|/\epsilon^{1-D}$ one hopes to capture more refined information about K , also relating to its finer topological structure.

In this paper we give sufficient conditions for L to exist, i.e., for $\epsilon^{-(d-D)}|K(\epsilon)|$ to converge to a limit as $\epsilon \rightarrow 0$, for self-similar sets $K \subset \mathbb{R}^d$; by self-similar here we mean both strictly self-similar and stochastically self-similar sets, i.e., random sets which are only self-similar at the level of distributions. As a matter of fact, for such K , we obtain exact asymptotics for $|K(\epsilon)|$ as $\epsilon \rightarrow 0$. It turns out that $\lim_{\epsilon \rightarrow 0} \epsilon^{-(d-D)}|K(\epsilon)|$ does *not* exist in many interesting cases, like for example the usual ternary Cantor set (see section 4.1). However, the limit $\lim_{T \rightarrow \infty} T^{-1} \int_0^T e^{(d-D)t} |K(e^{-t})| dt$ always exists for self-similar sets and one may use this quantity instead as a measure of lacunarity, if one is to summarize this information into a single number (compare with Bedford and Fisher [1]). We stress that our aim is to prove results on existence; the question of whether the proposed quantities are ‘good measures of lacunarity’ is a different issue, requiring separate investigation.

We remark in passing that the asymptotics of $|K(\epsilon)|$ are also directly related to the concept of ‘Minkowski measurability’. A compact set $K \subset \mathbb{R}^d$, of Minkowski dimension D , is Minkowski measurable precisely when $\epsilon^{-(d-D)}|K(\epsilon)|$ has a limit $L \in (0, \infty)$, as $\epsilon \rightarrow 0$; L is then the Minkowski content of K in this context. (Thus Mandelbrot defines lacunarity as the reciprocal of Minkowski content, when this is defined.) Minkowski measurability and Minkowski content have attracted interest in recent years, because of the central role they play in a conjecture of Lapidus ([14]), and related work of Lapidus and Pomerance ([15], [16]), pertaining to the

Weyl-Berry conjecture on the distribution of the eigenvalues of the Laplacian, on domains with ‘fractal’ boundaries.¹

Our main tool for studying the aforementioned asymptotics for $|K(\epsilon)|$ is the renewal theorem from probability theory and one of the objectives of this paper (besides obtaining the results themselves) is to emphasize the suitability of renewal theory as a tool in the study of self-similar sets. The first application of the (standard) renewal theorem to (strictly) self-similar sets is due to Lalley, who applied it in [12] to obtain the exact asymptotics for the packing and covering functions of such sets. Later, in [13], he used similar methods for fractals with some *approximate* self-similarity, such as limit sets of Kleinian groups; for the results of the latter paper he actually had to develop suitable analogues of the standard renewal theorem. We stress the fact that in both papers of Lalley mentioned above and also in section 2 of the present paper, pertaining to *strictly* self-similar sets, the renewal theorem is applied to study *deterministic* sets. One can use similar methods for *stochastically* self-similar sets and this is done in section 3 below. Results which hold in mean, for such random sets, are obtained by use of the ordinary renewal theorem, while almost sure results are obtained via an analogue of the renewal theorem for branching random walks, due to Nerman ([22]).

The rest of the paper is organized as follows. In section 2 we deal with strictly self-similar sets; the main results there are Theorem 2.3 and its companion, Theorem 2.4. In section 3 we deal with stochastically self-similar sets. In particular, in subsection 3.1 we describe the kind of random sets we are considering and state the main results concerning them, Theorems 3.1 and 3.2. In subsection 3.2 we state Nerman’s theorem in the language of branching random walks and in 3.3 we give the proof of Theorem 3.1. Subsection 3.4 deals with ‘mean convergence’ and in subsection 3.5 we prove Theorem 3.2. Finally, in section 4 we discuss some examples and applications. One of them pertains to level-sets of stable bridges and stable processes, for which $\epsilon^{-(d-D)}|K_t(\epsilon)|$ converges and the limit is local time at 0; here K_t is the zero-set of the process, run up to time t , and the convergence holds simultaneously for all $t \geq 0$, on a set of probability one (Theorem 4.1). We thus reprove a result which follows directly from results of Fristedt and Taylor ([9]). (Our proof, however, does not assume a priori existence of local time and we thus obtain local time as originally envisaged by P. Levy.)

After proving Theorem 2.3 (deterministic sets), we learned of Falconer’s paper [7], which also contains versions of Theorem 2.3 (i), but with the additional assumption that K is totally disconnected. He also uses renewal theorems, but his approach is, to a certain extent, different than ours. We decided to include Theorem 2.3 here because: 1) it covers basic examples, like e.g. the von Koch snowflake or the Sierpinski carpet, which Falconer’s result does not; 2) its proof conveys the basic idea of the approach better, as it lacks certain technicalities which are present in the random case.

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¹Note added in proof: After submitting the final version of this paper we also learned of [26]; conjectures 3 and 3r there relate directly to the results in this paper and the conjecture in section 5.

limit, as $\epsilon \rightarrow 0$, for the usual ternary Cantor set, and S. Lalley, who had independently obtained a renewal theorem for branching random walks, and from whom the author first heard about such a theorem.

2. STRICTLY SELF-SIMILAR SETS

2.1. Background and statement of results. Let $\{\phi_1, \dots, \phi_N\}$ be a finite set of similarities of $\mathbb{R}^d$; i.e., each ϕ_i satisfies

$$|\phi_i(x) - \phi_i(y)| = r_i |x - y| \quad \forall x, y \in \mathbb{R}^d,$$

for some constant r_i . Assume $0 < r_i < 1$, for each $i \in \{1, \dots, N\}$, i.e., each ϕ_i is a contraction, and let D be the unique real number for which

$$(2.1) \quad \sum_{i=1}^N r_i^D = 1.$$

A well known result of Hutchinson ([10]) asserts that there exists a unique nonempty compact subset K of $\mathbb{R}^d$ which is invariant with respect to $\{\phi_1, \dots, \phi_N\}$, i.e.,

$$(2.2) \quad K = \bigcup_{i=1}^N \phi_i(K).$$

The set K is then a (strictly) self-similar set and is in many cases a ‘fractal’. The examples in the introduction, which include the usual ternary Cantor set, the von Koch snowflake and the Sierpinski carpet, all arise in this manner (see [10]). We may, for example, obtain the ternary Cantor set by taking $\phi_1, \phi_2 : \mathbb{R} \rightarrow \mathbb{R}$, $\phi_1(x) = x/3$, $\phi_2(x) = x/3 + 2/3$. For the von Koch snowflake we may take $\phi_1, \dots, \phi_4 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where ϕ_i is the unique similarity, with positive determinant, mapping the interval $A_1 A_5$ to $A_i A_{i+1}$ in Figure 2. In both cases, K may then be visualized as (formally, the limit is in the Hausdorff metric)

$$K = \lim_{n \rightarrow \infty} \bigcup_{i_1, \dots, i_n} \phi_{i_1} \circ \dots \circ \phi_{i_n}([0, 1]).$$

Given a set of contracting similarities $\{\phi_1, \dots, \phi_N\}$, one says that $\{\phi_1, \dots, \phi_N\}$ (or the corresponding invariant set K) satisfies the *open set condition*, if there exists a bounded open set G in $\mathbb{R}^d$ with the properties

$$\begin{aligned} \phi_i(G) &\subseteq G \text{ for each } i \in \{1, \dots, N\}; \\ \phi_i(G) \cap \phi_j(G) &= \emptyset \text{ whenever } i \neq j. \end{aligned}$$

It is well known (see [10], [12], [21]) that if $\{\phi_1, \dots, \phi_N\}$ are contracting similarities, with contraction ratios $r_1, \dots, r_N$ respectively, satisfying the open set condition, then the corresponding invariant set K has both Hausdorff and Minkowski dimensions equal to the unique solution D of equation (2.1); this D is also referred

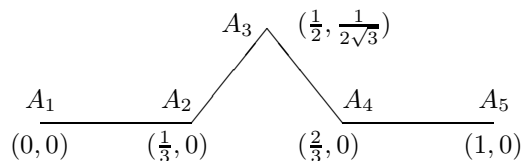


FIGURE 2. One step in the construction of the von Koch snowflake.

to as the similarity dimension of K . Furthermore, $0 < \mathcal{H}_D(K) < \infty$, where $\mathcal{H}_D$ denotes D -dimensional Hausdorff measure.

Notation 2.1. Given an arbitrary compact set $F \subset \mathbb{R}^d$ we will denote by $F(\epsilon)$ its ϵ -neighborhood,

$$F(\epsilon) := \{x \in \mathbb{R}^d : \text{dist}(x, F) < \epsilon\}.$$

Lebesgue measure—always d -dimensional in this paper, as the dimension of the underlying space—will be denoted by $|\cdot|$. With these notations $|K(\epsilon)|$ is the length, area, volume, etc. of the ϵ -neighborhood of K .

Notation 2.2. Given $\lambda > 0$ we write $\lambda\mathbb{Z} := \{\dots, -2\lambda, -\lambda, 0, \lambda, 2\lambda, \dots\}$.

It is well known that

$$\lim_{\epsilon \rightarrow 0} \frac{\log |K(\epsilon)|}{\log \epsilon} = d - D;$$

here D should be regarded as the Minkowski dimension of K . The following theorem is a refinement of this.

Theorem 2.3. *Let $\phi_1, \dots, \phi_N : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be contracting similarities, with similarity ratios $r_1, \dots, r_N \in (0, 1)$, and let K be the corresponding invariant set (2.2). Assume that $\{\phi_1, \dots, \phi_N\}$ satisfies the open set condition and let D satisfy $\sum_{i=1}^N r_i^D = 1$.*

- (i) *If the numbers $\log r_1, \dots, \log r_N$ do not concentrate on an additive subgroup of $\mathbb{R}$ of the form $\lambda\mathbb{Z}$ for some $\lambda > 0$, then the following limit exists and is finite:*

$$L := \lim_{\epsilon \rightarrow 0} \frac{|K(\epsilon)|}{\epsilon^{d-D}}.$$

- (ii) *The following limit always exists and is finite:*

$$L := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{|K(e^{-t})|}{e^{-t(d-D)}} dt.$$

Observe that the ternary Cantor set, the von Koch snowflake and the Sierpinski carpet are all excluded from (i).

Note. The number L is always defined, by (ii); when the limit in (i) exists it necessarily equals L .

Theorem 2.4. $L > 0$.

We defer the proof of Theorem 2.4 to section 2.4.

2.2. Proof of Theorem 2.3. Let $K_i = \phi_i(K)$, for $i \in \{1, \dots, N\}$, and observe that

$$\phi_i(K(\epsilon)) = K_i(\epsilon r_i);$$

here $K_i(\epsilon r_i)$ is the ϵr_i -neighborhood of $K_i = \phi_i(K)$. Since each ϕ_i is a similarity,

$$r_i^d |K(\epsilon)| = |\phi_i(K(\epsilon))| = |K_i(\epsilon r_i)|$$

or equivalently,

$$r_i^d \left| K \left(\frac{\epsilon}{r_i} \right) \right| = |K_i(\epsilon)|,$$

for any $\epsilon > 0$ and any $i \in \{1, \dots, N\}$. Let

$$(2.3) \quad R(\epsilon) = |K(\epsilon)| - \sum_{i=1}^N |K_i(\epsilon)|;$$

we then have that

$$(2.4) \quad \begin{aligned} |K(\epsilon)| &= \sum_{i=1}^N |K_i(\epsilon)| + R(\epsilon) \\ &= \sum_{i=1}^N r_i^d |K(\epsilon r_i^{-1})| + R(\epsilon). \end{aligned}$$

Notice here that in the totally disconnected case, i.e., when $K_i \cap K_j = \emptyset$ whenever $i \neq j$, there exists a $\delta > 0$ such that $\text{dist}(K_i, K_j) > \delta$ for $i \neq j$, and so $R(\epsilon) = 0$ for all sufficiently small ϵ . This is not true in the general case however, when distinct K_i may intersect.

Set

$$w(t) = \begin{cases} e^{(d-D)t} |K(e^{-t})| & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

and notice that (2.4) becomes

$$(2.5) \quad w(t) = \sum_{t + \log r_i \geq 0} r_i^D w(t + \log r_i) + z(t),$$

where for $t \geq 0$

$$(2.6) \quad z(t) = e^{(d-D)t} R(e^{-t}) + \sum_{t + \log r_i < 0} r_i^d e^{(d-D)t} |K(r_i^{-1} e^{-t})|$$

and $z(t) = 0$ for $t < 0$. Let F be the probability distribution function that puts mass r_i^D at the point $-\log r_i$. Then (2.5) is equivalent to the renewal equation

$$(2.7) \quad w(t) = z(t) + \int_{(0,t]} w(t-s) dF(s).$$

Observe that the function $R(e^{-t})$ is continuous (see (2.3)) and hence z has only finitely many discontinuities at the points $t = -\log r_i$, coming from the second term in (2.6). We also claim the following:

Lemma 2.5. *There exist constants $0 < C < \infty$ and $\delta > 0$ such that*

$$|z(t)| \leq C e^{-t\delta}.$$

The proof of the lemma is deferred to the next section.

By the lemma and the paragraph preceding it, the function z is directly Riemann-integrable and we may therefore apply the renewal theorem (see [8]). There are two cases to consider.

NON-LATTICE CASE: If the numbers $\log r_i$ are not all multiples of some $\lambda > 0$, then F is non-lattice and

$$(2.8) \quad \lim_{t \rightarrow \infty} w(t) = \frac{\int_0^\infty z(t) dt}{-\sum_{i=1}^N r_i^D \log r_i}.$$

This shows Theorem 2.3 (i).

To show (ii) in the non-lattice case observe that, since K is compact, the function w is, by definition, bounded on any finite interval and hence by (2.8)

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T w(t) dt = \frac{\int_0^\infty z(t) dt}{-\sum_{i=1}^N r_i^D \log r_i}.$$

LATTICE CASE: If there exists some number $\lambda > 0$ such that

$$\{\log r_1, \dots, \log r_N\} \subset \lambda \mathbb{Z}$$

and λ is the largest such number, then F is lattice with span λ and so the renewal theorem asserts that for each $s \in [0, \lambda)$

$$(2.9) \quad \lim_{n \rightarrow \infty} w(n\lambda + s) = \lambda \frac{\sum_{n=0}^\infty z(n\lambda + s)}{-\sum_{i=1}^N r_i^D \log r_i}.$$

We next turn to the proof of Theorem 2.3 (ii) in the lattice case. We begin by observing that w is bounded. This is because w is nonnegative, $\{w(n\lambda)\}_{n \geq 0}$ converges to a finite number, by (2.9) and Lemma 2.5, and, for any $t \geq 0$ and $s \in [0, \lambda)$,

$$w(t + s) = e^{(d-D)(t+s)} |K(e^{-t-s})| \leq e^{(d-D)(t+s)} |K(e^{-t})| \leq e^{(d-D)\lambda} w(t).$$

Next let $L(s)$ denote the limit on the right side of (2.9), for $s \in [0, \lambda)$. By the bounded convergence theorem and (2.9)

$$(2.10) \quad \lim_{n \rightarrow \infty} \int_{[0, \lambda)} w(n\lambda + s) ds = \int_{[0, \lambda)} L(s) ds.$$

Let $\lfloor \cdot \rfloor$ denote integer-part and write

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T w(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{\lfloor T/\lambda \rfloor - 1} \int_{(0, \lambda)} w(n\lambda + s) ds + \lim_{T \rightarrow \infty} \frac{1}{T} \int_{\lfloor T/\lambda \rfloor \lambda}^T w(t) dt.$$

The first term on the right here converges to $\lambda^{-1} \int_{[0, \lambda)} L(s) ds$, by (2.10) and Cesaro averaging, while the second term tends to 0 by the boundedness of w . Hence again

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T w(t) dt = \frac{1}{\lambda} \int_{(0, \lambda)} L(s) ds = \frac{\int_0^\infty z(t) dt}{-\sum_{i=1}^N r_i^D \log r_i}.$$

This proves (ii) in the lattice case and concludes the proof of the theorem. $\square$

2.3. Proof of Lemma 2.5. Recall equation (2.5) defining z . The second term on the right is nonnegative, bounded on any neighborhood of 0 and vanishes for $t > \max\{-\log r_1, \dots, -\log r_N\}$. Consequently, it suffices to show that there exist $0 < C < \infty$ and $\delta > 0$ such that

$$|R(\epsilon)| \leq C\epsilon^{d-D+\delta}$$

for all $0 < \epsilon < 1$. Now recall (2.3):

$$|R(\epsilon)| = \sum_{i=1}^N |K_i(\epsilon)| - |K(\epsilon)| \leq \sum_{i \neq j} |K_i(\epsilon) \cap K_j(\epsilon)|$$

and it is therefore enough to show that for some $0 < C < \infty$ and $\delta > 0$

$$(2.11) \quad |K_i(\epsilon) \cap K_j(\epsilon)| \leq C\epsilon^{d-D+\delta}$$

for all $i \neq j$ and $0 < \epsilon < 1$.

Towards this end fix $i \neq j$ and let $\epsilon > 0$. Call a subset F of K_i ϵ -separated if $x, y \in F \Rightarrow \text{dist}(x, y) \geq \epsilon$. Define $Q_{ij}(\epsilon)$ to be the maximum cardinality of an ϵ -separated subset F of K_i , such that $x \in F \Rightarrow \text{dist}(x, K_j) \leq \epsilon$. Then, by Proposition 3 of reference [12], there exist constants $0 < \gamma < \infty$ and $\delta > 0$, independent of ϵ , i and j , such that

$$(2.12) \quad Q_{ij}(\epsilon) \leq \gamma \epsilon^{\delta-D}.$$

The results of reference [12] require that $\{\phi_1, \dots, \phi_N\}$ satisfies the *strong* open set condition, i.e., that $K \cap G \neq \emptyset$ for the set G of the open set condition. It is now known however that the strong open set condition is equivalent to the open set condition (Schief [23]).

Now fix $0 < \epsilon < 1$ and let F be a maximal 2ϵ -separated subset of K_i , such that for each $z \in F$ we have $\text{dist}(z, K_j) \leq 2\epsilon$. Then, using the maximality of F ,

$$(2.13) \quad K_i(\epsilon) \cap K_j(\epsilon) \subseteq \bigcup_{y \in F} B_{3\epsilon}(y),$$

where $B_\rho(y)$ denotes the ball with center y and radius ρ . Since $\text{card } F = Q_{ij}(2\epsilon)$, we have then by (2.12) and (2.13)

$$\begin{aligned} |K_i(\epsilon) \cap K_j(\epsilon)| &\leq \text{card } F c_d (3\epsilon)^d \\ &\leq \gamma c_d 3^d 2^{\delta-D} \epsilon^{d-D+\delta} \\ &= C \epsilon^{d-D+\delta}, \end{aligned}$$

where $c_d = \frac{\Gamma(1/2)^d}{\Gamma(1+d/2)}$ is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. This shows (2.11) and completes the proof of Lemma 2.5. $\square$

2.4. Proof of Theorem 2.4. We will actually show that $\epsilon^{(d-D)}|K(\epsilon)|$ is bounded away from 0, for $0 < \epsilon \leq 1$.

Since $\{\phi_1, \dots, \phi_N\}$ satisfies the open set condition, it also satisfies the strong open set condition (cf. [23]); i.e., one can choose the open set G of the open set condition so that $K \cap G \neq \emptyset$. Fix an $x \in K \cap G \neq \emptyset$ and $\delta > 0$ such that $B_\delta(x) \subset G$.

For $0 < \epsilon \leq 1$, define

$$N(\epsilon) = \{(i_1 \dots i_n) \in \{1, \dots, N\}^* : r_{i_1} \dots r_{i_n} < \epsilon \leq r_{i_1} \dots r_{i_{n-1}}\}.$$

First note that since $\sum_{i=1}^N r_i^D = 1$ we also have

$$\sum_{(i_1 \dots i_n) \in N(\epsilon)} (r_{i_1} \dots r_{i_n})^D = 1$$

and as each term in the sum is $< \epsilon^D$, by the definition of $N(\epsilon)$,

$$(2.14) \quad \text{card } N(\epsilon) \geq \epsilon^{-D}.$$

We claim that

$$(2.15) \quad |K(\epsilon)| \geq c_d \epsilon^d \text{card } N(\epsilon/\delta r_{\min}),$$

where $r_{\min} := \min r_i$. To see this write $x_{i_1 \dots i_n} = \phi_{i_1} \circ \dots \circ \phi_{i_n}(x)$, where x is fixed above, and notice that, since $B_\delta(x) \subset G$,

$$B_{r_{i_1} \dots r_{i_n} \delta}(x_{i_1 \dots i_n}) \subset \phi_{i_1} \circ \dots \circ \phi_{i_n}(G).$$

If $i_1 \dots i_n$ and $j_1 \dots j_m$ are two distinct sequences in $N(\epsilon/\delta r_{\min})$, then $i_p \neq j_p$ for some $p \leq \min\{m, n\}$. This implies then that

$$\phi_{i_1} \circ \dots \circ \phi_{i_n}(G) \cap \phi_{j_1} \circ \dots \circ \phi_{j_m}(G) = \emptyset$$

and therefore

$$B_{r_{i_1} \dots r_{i_n} \delta}(x_{i_1 \dots i_n}) \cap B_{r_{j_1} \dots r_{j_m} \delta}(x_{i_1 \dots i_n}) = \emptyset.$$

But for any sequence $(i_1 \dots i_n) \in N(\epsilon/\delta r_{\min})$ we have

$$r_{i_1} \dots r_{i_n} \delta \geq \frac{\epsilon}{\delta r_{\min}} r_{i_n} \delta \geq \epsilon,$$

so the balls $B_\epsilon(x_{i_1 \dots i_n})$, $(i_1 \dots i_n) \in N(\epsilon/\delta r_{\min})$, are disjoint. Inequality (2.15) now follows.

Combining (2.14) and (2.15) we have that

$$\epsilon^{(d-D)} |K(\epsilon)| \geq c_d \delta^D r_{\min}^D > 0,$$

for all $0 < \epsilon \leq 1$. □

3. STOCHASTICALLY SELF-SIMILAR SETS

3.1. Background and statement of main result. In this section we describe the kind of random sets we are considering and state our main result concerning them. The construction below follows Mauldin and Williams ([20]) and can also be found in Falconer ([6]).

Let J be a nonempty compact set in $\mathbb{R}^d$ and write $G := \text{int}(J)$. We assume that $\overline{G} = J$. Let $\mathcal{S}$ denote the set of all contracting similarities of $\mathbb{R}^d$. We assume as given a probability space, on which we have defined a random element Φ , taking either the value $\mathbf{e}$ (for empty) or $\Phi = \{\phi_1, \dots, \phi_\nu\}$, where ν is an integer-valued random variable ($\nu = 0$ if $\Phi = \mathbf{e}$) and each ϕ_i is in $\mathcal{S}$, and furthermore

$$\begin{aligned} \phi_i(J) &\subseteq J \text{ for each } i = 1, \dots, \nu; \\ \phi_i(G) \cap \phi_j(G) &= \emptyset \text{ whenever } i \neq j, 1 \leq i, j \leq \nu. \end{aligned}$$

(See the Appendix for matters of measurability.) We further write r for the constant ratio $|\phi(x) - \phi(y)|/|x - y|$, $x \neq y$, for a similarity ϕ ; r and ϕ will always be in this relation in the sequel. Note that $\phi \in \mathcal{S} \Rightarrow r \in (0, 1)$.

The random set of interest is defined by means of the family tree of a Galton-Watson branching process. Let $\mathbb{N}^* = \{o\} \cup \bigcup_{n=1}^{\infty} \mathbb{N}^n$ (o for ‘ancestor’) and for each $x \in \mathbb{N}^*$ let $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ be a copy of the above probability space. Set $(\Omega, \mathcal{F}, \mathbb{P}) = \prod_{x \in \mathbb{N}^*} (\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$. On $(\Omega, \mathcal{F}, \mathbb{P})$ we then have defined independent and identically distributed random elements $\Phi_x = \{\phi_{x1}, \dots, \phi_{x\nu_x}\}$, $x \in \mathbb{N}^*$ ($\Phi_x = \mathbf{e}$ if $\nu_x = 0$).

Write $\mathcal{G}_0 = \{o\}$ and inductively define $\mathcal{G}_{n+1} = \emptyset$, if $\mathcal{G}_n = \emptyset$, and

$$\mathcal{G}_{n+1} = \{(x, i) : x \in \mathcal{G}_n, \nu_x \neq 0, 1 \leq i \leq \nu_x\},$$

if $\mathcal{G}_n \neq \emptyset$; observe that $\mathcal{G}_{n+1} = \emptyset$ if $\nu_x = 0$ for all $x \in \mathcal{G}_n$. For each $n \geq 0$, $\mathcal{G}_n$ represents the individuals in the n -th generation of a Galton-Watson branching process with mean family size $\mathbb{E}[\nu] = \mathbb{E}[\text{card } \mathcal{G}_1]$. Write

$$\mathcal{P} := \bigcup_{n=0}^{\infty} \mathcal{G}_n$$

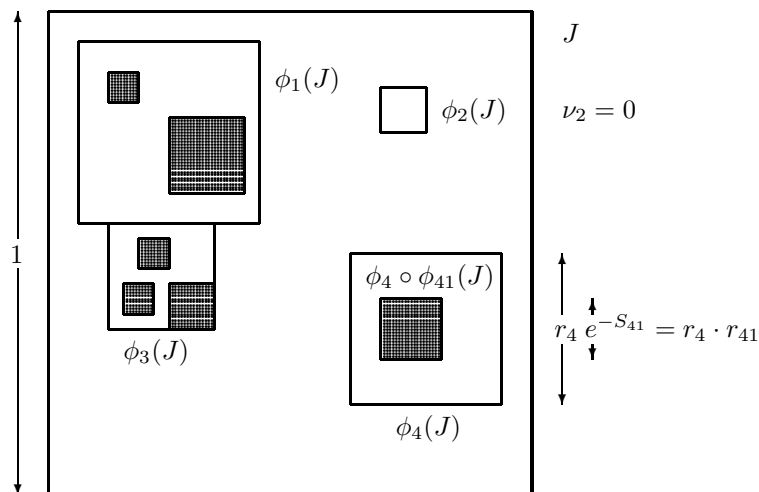


FIGURE 3. Two steps in the construction of a random set.

for the entire population. Writing $x = x_1 \dots x_n$ for $x \in \mathcal{G}_n$, the set

$$K := \bigcap_{n=1}^{\infty} \bigcup_{x \in \mathcal{G}_n} \phi_{x_1} \circ \phi_{x_1 x_2} \circ \dots \circ \phi_{x_1 \dots x_n}(J)$$

defines then a random compact set in $\mathbb{R}^d$; K is a stochastically self-similar set.

It is well known (see for example [20]) that $\mathbb{P}(K \neq \emptyset) > 0$ iff $\mathbb{E}[\nu] > 1$; we will therefore assume throughout that

$$(3.1) \quad 1 < \mathbb{E}[\nu] < \infty.$$

We now identify a branching random walk naturally associated with the construction, which will be key in the sequel. Set $S_0 = 0$ and for $n \geq 1$ inductively define

$$S_{x_1 \dots x_n} = S_{x_1 \dots x_{n-1}} + \log r_{x_1 \dots x_n}^{-1},$$

where, recall, r_x stands for the constant factor of contraction of the similarity ϕ_x . Note that S_x is well defined whenever $x \in \mathcal{P}$; for notational convenience set $S_x = \infty$ for $x \in \mathbb{N}^* \setminus \mathcal{P}$. $\{S_x : x \in \mathbb{N}^*\}$ is then a branching random walk with positive step-sizes (since each ϕ_x is a contraction).

For $\theta \in \mathbb{R}$, set

$$m(\theta) := \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} e^{-\theta S_x} \right] = \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} r_x^\theta \right],$$

$$m'(\theta) := \mathbb{E} \left[- \sum_{x \in \mathcal{G}_1} S_x e^{-\theta S_x} \right].$$

By our assumption (3.1) we have $m(\theta) < \infty$ for all $\theta \geq 0$ and $-m'(\theta) < \infty$ for all $\theta > 0$. The function m is a decreasing convex C^∞ function on $(0, \infty)$, which is continuous on $[0, \infty)$, with $m(0) = \mathbb{E}[\nu] > 1$ and $m(\infty) = 0$. Consequently, there exists a unique number $D > 0$ such that $m(D) = 1$. By a well known result, obtained independently by Mauldin and Williams ([20]) and Falconer ([6]), the

number D is then the Hausdorff dimension of the random set K , almost surely on the event $\{K \neq \emptyset\}$. That it is also the Minkowski dimension of K (on $\{K \neq \emptyset\}$) follows easily from the results of the present paper and is a well known fact.

We now state the main result of this paper for stochastically self-similar sets. Observe that the random set K is, probabilistically, uniquely determined by the compact set J and the random set of similarities $\Phi = \{\phi_1, \dots, \phi_\nu\}$. Also recall that $G = \text{int}(J)$, that $F(\epsilon)$ denotes the ϵ -neighborhood of a compact set F (Notation 2.1) and that $|\cdot|$ denotes d -dimensional Lebesgue measure.

Theorem 3.1. *Assume of Φ and J that $1 < \mathbb{E}[\nu] < \infty$ and that*

$$(3.2) \quad \mathbb{P}(K \cap G \neq \emptyset) > 0.$$

- (i) *Assume that the measure $A \mapsto \mathbb{E}[\sum_{x \in \mathcal{G}_1} \mathbf{1}_A(S_x)]$ is non-lattice; i.e., it does not concentrate on an additive subgroup of $\mathbb{R}$ of the form $\lambda\mathbb{Z}$ for some $\lambda > 0$. Then the following limit exists and is finite a.s.:*

$$L := \lim_{\epsilon \rightarrow 0} \frac{|K(\epsilon)|}{\epsilon^{d-D}}.$$

- (ii) *The following limit always exists and is finite a.s.:*

$$L := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{(d-D)t} |K(e^{-t})| dt.$$

Note. As in the deterministic case, the random variable L is always defined by part (ii) of the above theorem. When the limit in (i) exists, it necessarily equals the limit in (ii); hence there is no ambiguity in using the same letter for both limits.

Theorem 3.2. *Set $M_1 = \sum_{x \in \mathcal{G}_1} r_x^D$.*

- (i) *If $\mathbb{E}[M_1 \log^+ M_1] = \infty$, then $L = 0$ with probability 1.*
(ii) *If $\mathbb{E}[M_1 \log^+ M_1] < \infty$, then $\mathbb{P}(L > 0 | K \neq \emptyset) = 1$.*

Remark. Note that $\mathbb{E}[\nu \log^+ \nu] < \infty$ implies $\mathbb{E}[M_1 \log^+ M_1] < \infty$.

We defer the proof of Theorem 3.1 to section 3.3 and that of Theorem 3.2 to section 3.5.

3.2. Renewal theorem for branching random walks. Let $\{S_x : x \in \mathbb{N}^*\}$ be the branching random walk associated with the construction of K and recall that this walk has positive steps. Denote by ξ the random measure (point process) $\xi(A) := \sum_{x \in \mathcal{G}_1} \mathbf{1}_A(S_x)$, for Borel sets A in $\mathbb{R}$. Since the walk has positive steps, ξ is concentrated on $(0, \infty)$.

Recall that D is the unique solution to the equation $\mathbb{E}[\sum_{x \in \mathcal{G}_1} e^{-DS_x}] = 1$ and set

$$M_n := \sum_{x \in \mathcal{G}_n} e^{-DS_x}, \quad n \geq 0.$$

A straightforward computation shows that the sequence $(M_n)_{n \geq 0}$ is a non-negative martingale (relative to the natural filtration associated with the successive steps of the construction of K). Denote

$$(3.3) \quad M_\infty := \lim_{n \rightarrow \infty} M_n.$$

The following theorem is due to Biggins ([3]); see Lyons [17] for a conceptual proof, following work by Lyons, Pemantle and Peres [18].

Theorem 3.3 (Biggins). *The following are equivalent:*

- (i) $\mathbb{E}[M_1 \log^+ M_1] < \infty$;
- (ii) $\mathbb{P}(M_\infty = 0) < 1$;
- (iii) $\mathbb{P}(M_\infty > 0 \mid \text{non-extinction}) = 1$;
- (iv) $\mathbb{E}[M_\infty] = 1$;
- (v) $M_n \longrightarrow M_\infty$ in L^1 .

Recall that our underlying probability space $(\Omega, \mathcal{F}, \mathbb{P}) = \prod_{x \in \mathbb{N}^*} (\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$ is a product-space. For each $x \in \mathbb{N}^*$ let $\sigma_x : \Omega \rightarrow \Omega$ be the shift, defined by

$$(\sigma_x \omega)_y = \omega_{xy},$$

where, if $x = x_1 \dots x_n$ and $y = y_1 \dots y_m$, then $xy = x_1 \dots x_n y_1 \dots y_m$.

Let $Z = \{Z_t : t \in \mathbb{R}\}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, which is jointly measurable in (ω, t) and is separable and takes values in the space of functions which possess both left- and right-hand limits and vanish on $(-\infty, 0)$. Given such a process Z , write $Z^x = \{Z_t^x : t \in \mathbb{R}\}$ for the process

$$Z_t^x(\omega) = Z_t(\sigma_x \omega),$$

for $x \in \mathbb{N}^*$. So if Z is some process associated with the branching random walk $\{S_x : x \in \mathbb{N}^*\}$ (or the construction leading to K), then Z^x is the corresponding process associated with that part of the population which emanates from individual x . The following theorem is due to Nerman [22] and is an analogue of the renewal theorem for ordinary random walks.

Theorem 3.4. *Suppose that there exists a non-increasing and integrable function $h : [0, \infty) \rightarrow (0, \infty)$, such that*

$$(3.4) \quad \mathbb{E} \left[\sup_{t \geq 0} \frac{e^{-Dt} |Z_t|}{h(t)} \right] < \infty.$$

- (i) [Nerman] *If the measure $A \mapsto \mathbb{E}[\xi(A)]$ is non-lattice, then a.s., as $t \longrightarrow \infty$,*

$$e^{-Dt} \sum_{x \in \mathcal{P}} Z_{t-S_x}^x \longrightarrow \frac{M_\infty}{-m'(D)} \int_0^\infty e^{-Ds} \mathbb{E}[Z_s] ds.$$

- (ii) *If the measure $A \mapsto \mathbb{E}[\xi(A)]$ is lattice, with span $\lambda > 0$, then for every $s \in [0, \lambda)$, we have that a.s., as $n \longrightarrow \infty$,*

$$e^{-Dn\lambda} \sum_{x \in \mathcal{P}} Z_{s+n\lambda-S_x}^x \longrightarrow \frac{\lambda M_\infty}{-m'(D)} \sum_{n=0}^\infty e^{-Dn\lambda} \mathbb{E}[Z_{n\lambda+s}].$$

Remark. Recall our standing assumption $1 < \mathbb{E}[\nu] < \infty$. Nerman's theorem actually requires less than $\mathbb{E}[\nu] < \infty$; see [22] for the precise conditions needed. Furthermore, the lattice case does *not* require Z to have sample paths which possess right- and left-hand limits.

Proof. Assertion (i) is Theorem 5.4 of Nerman [22].

(ii) A close examination of Nerman's proof reveals that the non-lattice assumption is only used in two places: on page 384, *Proof of Theorem 5.4 from Corollary 5.11* and in the second part of the *Proof of Lemma 5.10* (establishing (5.53)). Of these the first is irrelevant for the lattice case and his proof of Lemma 5.10 (with obvious modifications) yields the following statement for the lattice case: *If $p \in \mathbb{N}$ is*

such that $\mathbb{E}[\xi(0, p\lambda)] > 1$ and (3.4) is satisfied (and given our standing assumption $1 < \mathbb{E}[\nu] < \infty$), then, for any $t \geq 0$,

$$(3.5) \quad e^{-Dkp\lambda} \sum_{x \in \mathcal{P}} Z_{t+kp\lambda-S_x}^x \longrightarrow \frac{\lambda M_\infty}{-m'(D)} \sum_{n=-\infty}^{\infty} e^{-Dn\lambda} \mathbb{E}[Z_{t+n\lambda}]$$

as $k \rightarrow \infty$, on a set B_t with $\mathbb{P}(B_t) = 1$.

Fix $s \in [0, \lambda)$ and set $A_s = B_s \cap B_{s+\lambda} \cap \dots \cap B_{s+(p-1)\lambda}$. Then $\mathbb{P}(A_s) = 1$. On A_s , and given $\epsilon > 0$, choose k_ϵ so that, for $k \geq k_\epsilon$, $e^{-Dkp\lambda} \sum_{x \in \mathcal{P}} Z_{t+kp\lambda-S_x}^x$ is within $\pm\epsilon$ from its limit for all $t \in \{s, s+\lambda, \dots, s+(p-1)\lambda\}$. For an arbitrary $n \in \mathbb{N}$ write $n = k(n)p + q$, $0 \leq q < p$. Then, if $n \geq k_\epsilon p + p - 1$,

$$e^{-Dn\lambda} \sum_{x \in \mathcal{P}} Z_{s+n\lambda-S_x}^x = e^{-Dq\lambda} e^{-Dk(n)p\lambda} \sum_{x \in \mathcal{P}} Z_{s+q\lambda+k(n)p\lambda-S_x}^x$$

is within $\pm\epsilon$ from its limit

$$e^{-Dq\lambda} \frac{\lambda M_\infty}{-m'(D)} \sum_{m=-\infty}^{\infty} e^{-Dm\lambda} \mathbb{E}[Z_{s+q\lambda+m\lambda}] = \frac{\lambda M_\infty}{-m'(D)} \sum_{m=-\infty}^{\infty} e^{-Dm\lambda} \mathbb{E}[Z_{s+m\lambda}].$$

This proves the convergence asserted in (ii). $\square$

We close this section with a simple but useful fact. The sequence

$$M_n^{(\theta)} := [m(\theta)]^{-n} \sum_{x \in \mathcal{G}_n} e^{-\theta S_x}, \quad n \geq 0,$$

is a nonnegative martingale, for each $\theta \geq 0$, as then $m(\theta) < \infty$. Hence $M_n^{(\theta)}$ converges to a finite limit $M_\infty^{(\theta)}$. For $\theta > D$ we have $0 < m(\theta) < 1$ and therefore $\sum_{x \in \mathcal{G}_n} e^{-\theta S_x} \rightarrow 0$, as $n \rightarrow \infty$. Consequently, with probability equal to one,

$$(3.6) \quad \min_{x \in \mathcal{G}_n} S_x \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

3.3. Proof of Theorem 3.1. Recall that $|K(\epsilon)|$ denotes the ϵ -neighborhood of the random set K . Then

$$|K(\epsilon)| = \sum_{x \in \mathcal{G}_1} |[K \cap \phi_x(J)](\epsilon)| - Q(\epsilon).$$

Let $K_x = K \cap \phi_x(J)$ and $K^x = \phi_x^{-1}(K \cap \phi_x(J))$; observe that the set K^x has the same distribution as K and that the K^x , $x \in \mathcal{G}_1$, are independent (given the σ -algebra $\sigma(\Phi_0)$). Since each ϕ_x is a similarity,

$$|K_x(\epsilon)| = e^{-dS_x} |K^x(\epsilon e^{S_x})|$$

and so, with

$$(3.7) \quad R_t = Q(e^{-t}) \mathbf{1}_{[0, \infty)}(t)$$

and

$$W_t = e^{(d-D)t} |K(e^{-t})| \mathbf{1}_{[0, \infty)}(t),$$

we have that

$$(3.8) \quad W_t = \sum_{x \in \mathcal{G}_1} e^{-dS_x} W_{t-S_x}^x + e^{-Dt} Z_t,$$

where, for $t \geq 0$,

$$(3.9) \quad Z_t := \sum_{x \in \mathcal{G}_1} e^{d(t-S_x)} |K^x(e^{-t+S_x})| \mathbf{1}_{(t,\infty)}(S_x) - e^{dt} R_t$$

and $Z_t := 0$ for $t < 0$. Then, upon iterating (3.8), we obtain

$$(3.10) \quad W_t = e^{-Dt} \sum_{x \in \mathcal{P}} Z_{t-S_x}^x.$$

Observe that the process $Z = \{Z_t : t \in \mathbb{R}\}$ satisfies $Z_t = 0$ for $t < 0$ and has paths which are left-continuous with right-hand limits; as a matter of fact Z is continuous except possibly at the points S_x , $x \in \mathcal{G}_1$. We shall apply Theorem 3.4 and to do so we need to verify that condition (3.4) holds. We will do this in two separate lemmata whose proofs we postpone to the end of this section.

Lemma 3.5. *There exists a non-increasing and integrable function $g : [0, \infty) \rightarrow (0, \infty)$, for which*

$$\mathbb{E} \left[\sup_{t \geq 0} \frac{e^{-Dt}}{g(t)} \sum_{x \in \mathcal{G}_1} e^{d(t-S_x)} |K^x(e^{-t+S_x})| \mathbf{1}_{(t,\infty)}(S_x) \right] < \infty.$$

Lemma 3.6. *There exists a non-increasing and integrable function $h : [0, \infty) \rightarrow (0, \infty)$, for which*

$$\mathbb{E} \left[\sup_{t \geq 0} \frac{e^{(d-D)t} R_t}{h(t)} \right] < \infty.$$

It follows from (3.9) and the two lemmata that

$$(3.11) \quad \mathbb{E} \left[\sup_{t \geq 0} \frac{e^{-Dt} |Z_t|}{h(t) + g(t)} \right] < \infty,$$

with $h + g$ non-increasing and integrable.

In view of (3.11) and the paragraph preceding Lemma 3.5, we may now apply Theorem 3.4 to (3.10).

(i) Assume that the measure $A \mapsto \mathbb{E}[\xi(A)]$ is non-lattice. Then by Theorem 3.4 we have that with probability equal to one, as $t \rightarrow \infty$,

$$(3.12) \quad W_t \rightarrow \frac{M_\infty}{-m'(D)} \int_0^\infty e^{-Ds} \mathbb{E}[Z_s] ds.$$

This shows Theorem 3.1 (i).

(ii) Observe that, because $T \mapsto T^{-1} \int_0^T W_t(\omega) dt$ is continuous, for each ω , the functions

$$\omega \mapsto \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T W_t(\omega) dt \quad \text{and} \quad \omega \mapsto \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T W_t(\omega) dt$$

are measurable and therefore the set of ω for which $\lim_{T \rightarrow \infty} T^{-1} \int_0^T W_t(\omega) dt$ exists is measurable. It therefore suffices to show that this set of convergence contains a set of probability one.

Assume first $A \mapsto \mathbb{E}[\xi(A)]$ is non-lattice. From (i) we have that

$$W_t = e^{(d-D)t} |K(e^{-t})| \rightarrow L, \quad \text{a.s.},$$

as $t \rightarrow \infty$. As $W_t(\omega)$ is bounded on finite intervals (actually uniformly in ω , by compactness of J), it follows that, as $T \rightarrow \infty$,

$$\frac{1}{T} \int_0^T W_t dt \rightarrow L,$$

on the event $\{\omega : W_t(\omega) \rightarrow L(\omega)\}$, i.e., on a set of probability one.

We next turn to the lattice case. Assume that the measure $A \mapsto \mathbb{E}[\xi(A)]$ is concentrated on a lattice $\lambda\mathbb{Z}$ and that λ is the largest such number.

By Theorem 3.4 (ii), for each $s \in [0, \lambda)$, there exists a set A_s of probability one, such that, on A_s , as $n \rightarrow \infty$,

$$W_{s+n\lambda} \rightarrow \lambda e^{-Ds} \frac{M_\infty}{-m'(D)} \sum_{k=0}^{\infty} e^{-Dk\lambda} \mathbb{E}[Z_{s+k\lambda}].$$

Call the limit on the right-hand side L_s and set $A := \bigcap_r A_r$, the intersection being over all rational r in $[0, \lambda)$. Then $\mathbb{P}(A) = 1$ and

$$(3.13) \quad \lim_{n \rightarrow \infty} W_{r+n\lambda}(\omega) = L_r(\omega), \quad \text{for all } \omega \in A \text{ and } r \in \mathbb{Q} \cap [0, \lambda).$$

For given $s \in [0, \lambda)$ have sequences $\{r_n\}$ and $\{q_n\}$ in $\mathbb{Q} \cap [0, \lambda)$ such that $r_n \uparrow s$ and $q_n \downarrow s$. Since for $t, s \geq 0$ one has $K(e^{-t-s}) \subseteq K(e^{-t})$, it follows that

$$W_{t+s} \leq e^{(d-D)s} W_t \quad (t, s \geq 0).$$

Using this and (3.13) we obtain that, for all $\omega \in A$ and all $k \geq 1$,

$$(3.14) \quad \begin{aligned} e^{-(d-D)(q_k-s)} L_{q_k}(\omega) &\leq \liminf_{n \rightarrow \infty} W_{s+n\lambda}(\omega) \\ &\leq \limsup_{n \rightarrow \infty} W_{s+n\lambda}(\omega) \leq e^{(d-D)(s-r_k)} L_{r_k}(\omega). \end{aligned}$$

Next observe that the function $t \mapsto \mathbb{E}[Z_t]$ has left- and right-hand limits. This is so because the sample-paths of Z have this property and we can use the dominated convergence theorem by (3.11). Consequently, the set C where this function is continuous has full Lebesgue measure; as a matter of fact C^c is countable. Using (3.11) again, we conclude that the function

$$t \mapsto \sum_{k=0}^{\infty} e^{-Dk\lambda} \mathbb{E}[Z_{t+k\lambda}], \quad t \in [0, \lambda),$$

is continuous outside $\bigcup_{k=0}^{\infty} (C^c - k\lambda)$, a countable set. Thus for Lebesgue-a.a. $s \in [0, \lambda)$ we have that $L_{q_k}(\omega) \rightarrow L_s(\omega)$ and $L_{r_k}(\omega) \rightarrow L_s(\omega)$, as $k \rightarrow \infty$ (recall the definition of L_s), and so by (3.14),

$$W_{s+n\lambda}(\omega) \rightarrow L_s(\omega) \quad (n \rightarrow \infty)$$

for all $\omega \in A$. Just as in the deterministic case now, we can conclude that

$$(3.15) \quad \frac{1}{T} \int_0^T W_t(\omega) dt \rightarrow \frac{1}{\lambda} \int_{[0, \lambda)} L_s(\omega) ds = \frac{M_\infty(\omega)}{-m'(D)} \int_0^\infty e^{-Ds} \mathbb{E}[Z_s] ds,$$

as $T \rightarrow \infty$, for all $\omega \in A$, i.e., on a set of probability one. This proves Theorem 3.1 (ii) in the lattice case and completes the proof of the theorem. $\square$

Proof of Lemma 3.5. Recall that c_d denotes the Lebesgue measure of the unit ball in $\mathbb{R}^d$ and observe that, for any $x \in \mathbb{N}^*$,

$$(3.16) \quad |K^x(\epsilon)| \leq |J(\epsilon)| \leq c_d [\text{diam}(J) + \epsilon]^d.$$

Hence, if $S_x > t$,

$$e^{d(t-S_x)} |K^x(e^{-t+S_x})| \leq c_d \sup_{\epsilon > 1} \left(\frac{\text{diam}(J) + \epsilon}{\epsilon} \right)^d = c_d [\text{diam}(J) + 1]^d.$$

It follows that, for $g(t) = e^{-Dt} \mathbf{1}_{[0, \infty)}(t)$,

$$\mathbb{E} \left[\sup_{t \geq 0} \frac{e^{-Dt}}{g(t)} \sum_{x \in \mathcal{G}_1} e^{d(t-S_x)} |K^x(e^{-t+S_x})| \mathbf{1}_{(t, \infty)}(S_x) \right] \leq c_d [\text{diam}(J) + 1]^d \mathbb{E}[\nu] < \infty. \quad \square$$

Proof of Lemma 3.6. The proof is a probabilistic analogue of the proof of the key estimate of reference [12]. It is rather long and we break it up into several steps.

Assume (3.2), that is,

$$\mathbb{P}(K \cap G \neq \emptyset) > 0.$$

For $\alpha > 0$ and $p \in \mathbb{N}$ let

$$\mathcal{G}_p(\alpha) = \{x \in \mathcal{G}_p : \exists v \in \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_p}(J) \text{ with } \text{dist}(v, G^c) \leq \alpha\}.$$

Claim 1. There exist $p \in \mathbb{N}$ and $\alpha > 0$ such that $\mathbb{P}(\mathcal{G}_p \setminus \mathcal{G}_p(\alpha) \neq \emptyset) > 0$.

Proof. Let $\partial\mathcal{P}$ consist of all infinite sequences $x = x_1 x_2 \dots \in \mathbb{N}^{\mathbb{N}}$ for which

$$x_1 \dots x_n \in \mathcal{G}_n \quad \text{for all } n \geq 1.$$

By (3.6) and nestedness, we have that $\bigcap_{n \geq 1} \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_n}(J)$ is a singleton for each $x \in \partial\mathcal{P}$, whose unique element we denote by v_x ; so $K = \{v_x : x \in \partial\mathcal{P}\}$.

Suppose now that $\mathbb{P}(\mathcal{G}_p = \mathcal{G}_p(\alpha)) = 1$, for all $p \in \mathbb{N}$ and $\alpha > 0$. Then $\Omega' = \bigcap_{\alpha \in \mathbb{Q}^+} \bigcap_{p \geq 1} \{\mathcal{G}_p = \mathcal{G}_p(\alpha)\}$ has probability one. Now either $K = \emptyset$ or if $v \in K$, then $v = v_x$ for some $x \in \partial\mathcal{P}$; then $v_x \in \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_n}(J)$ for all $n \geq 1$. On Ω' we have $x_1 \dots x_n \in \mathcal{G}_n(\alpha)$ for all $n \geq 1$ and all $\alpha \in \mathbb{Q}^+$ and so

$$\text{dist}(v_x, G^c) \leq e^{-S_{x_1 \dots x_n}} \text{diam}(J) + \alpha,$$

which, together with (3.6) again, implies that $v_x \in G^c$. This shows that $K \cap G = \emptyset$ on Ω' , creating a contradiction. $\square$

From now on we fix a p and an α as in Claim 1. Let

$$\tilde{\mathcal{G}}_0 = \mathcal{G}_0, \dots, \tilde{\mathcal{G}}_{p-1} = \mathcal{G}_{p-1}, \tilde{\mathcal{G}}_p = \mathcal{G}_p(\alpha)$$

and for $n > p$ let $\tilde{\mathcal{G}}_n$ consist of those $x = x_1 \dots x_n \in \mathcal{G}_n$ which have the following property: for each $1 \leq i \leq n - p + 1$ there exists $v \in \phi_{x_1 \dots x_i} \circ \dots \circ \phi_{x_1 \dots x_{i+p-1}}(J)$ with $\text{dist}(v, G^c) \leq \alpha$. Set also $\tilde{\mathcal{P}} = \bigcup_{n \geq 0} \tilde{\mathcal{G}}_n$.

For $t \geq 0$ let $\mathcal{P}_t$ consist of those $x = x_1 \dots x_m \in \mathcal{P}$ for which

$$S_{x_1 \dots x_{m-1}} \leq t < S_{x_1 \dots x_m};$$

for $t < 0$ let $\mathcal{P}_t = \{\emptyset\}$ and define $\tilde{\mathcal{P}}_t$ accordingly.

For $\epsilon > 0$, let $Q(\epsilon)$ be the maximum cardinality of an ϵ -separated subset F of K , such that $\text{dist}(v, G^c) \leq \epsilon$ for all $v \in F$. Finally, set $s := \log \alpha^{-1} + \log(\text{diam}(J))$.

Claim 2. With $t = \log \epsilon^{-1} + \log(\text{diam}(J))$, we have that for all $0 < \epsilon \leq \alpha$

$$Q(\epsilon) \leq \sum_{x \in \tilde{\mathcal{P}}_{t-s}} \text{card } \mathcal{P}_{t-S_x}^x.$$

Proof. Let F be an ϵ -separated subset of K , such that $\text{dist}(v, G^c) \leq \epsilon$ for all $v \in F$. By the representation $K = \{v_x : x \in \partial\mathcal{P}\}$ (see the proof of Claim 1), to each point $v \in F$ corresponds an element $x = x_1 \dots x_m \in \mathcal{P}_t$, and different elements of F have different representatives in $\mathcal{P}_t$, because F is ϵ -separated. Now given $x = x_1 \dots x_m \in \mathcal{P}_t$ let $q = q_x$ be the unique integer in $\{1, \dots, m\}$ for which

$$S_{x_1 \dots x_{q-1}} \leq \log\left(\frac{\alpha}{\epsilon}\right) < S_{x_1 \dots x_q}.$$

We claim that if $x = x_1 \dots x_m \in \mathcal{P}_t$ corresponds to an element $v_x \in F$, then $x_1 \dots x_q \in \tilde{\mathcal{G}}_q$. For if not, then for some $1 \leq i \leq q - p + 1$ we would have that $\text{dist}(u, G^c) > \alpha$, for all $u \in \phi_{x_1 \dots x_i} \circ \dots \circ \phi_{x_1 \dots x_{i+p-1}}(J)$, so

$$\begin{aligned} & \text{dist}(\phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_{i+p-1}}(u), \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_{i-1}}(G^c)) \\ &= \exp(-S_{x_1 \dots x_{i-1}}) \text{dist}(\phi_{x_1 \dots x_i} \circ \dots \circ \phi_{x_1 \dots x_{i+p-1}}(u), G^c) \\ &> \exp(-S_{x_1 \dots x_{i-1}}) \alpha \\ &\geq \exp(-S_{x_1 \dots x_{q-1}}) \alpha \geq \epsilon, \end{aligned}$$

for all $u \in J$; equivalently,

$$(3.17) \quad \text{dist}(u, \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_{i-1}}(G^c)) > \epsilon,$$

for all $u \in \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_{i+p-1}}(J)$ and so in particular (3.17) would be true for $u = v_x$. But as $G^c \subseteq \phi_{x_1} \circ \dots \circ \phi_{x_1 \dots x_{i-1}}(G^c)$ we then would have that $\text{dist}(v_x, G^c) > \epsilon$, contradicting the fact that $v_x \in F$.

It now follows from the above that for $0 < \epsilon \leq \alpha$

$$Q(\epsilon) \leq \sum_{x \in \mathcal{P}_t} \mathbf{1}_{\tilde{\mathcal{G}}_q}(x_1 \dots x_q) = \sum_{x \in \mathcal{P}_{t-s}} \text{card } \mathcal{P}_{t-S_x}^x. \quad \square$$

Now let $\tilde{D}$ be the unique solution to

$$(3.18) \quad \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_p} e^{-\tilde{D} S_x} \right] = 1;$$

because $\mathbb{P}(\mathcal{G}_p \setminus \tilde{\mathcal{G}}_p \neq \emptyset) > 0$, by our choice of p and α , we must have $\tilde{D} < D$.

Claim 3. For any $\delta > 0$ we have

$$\mathbb{E} \left[\sup_{0 < \epsilon \leq \alpha} \epsilon^{\tilde{D} + \delta} Q(\epsilon) \right] < \infty.$$

Proof. Assume first that $p = 1$. Then $\{\tilde{\mathcal{G}}_n : n \geq 0\}$ corresponds to a Galton-Watson branching process, just like $\{\mathcal{G}_n : n \geq 0\}$, and $\{S_x : x \in \tilde{\mathcal{P}}\}$ is a branching random walk, just like $\{S_x : x \in \mathcal{P}\}$, with different parameters however.

Recall the random measure $\xi(A) = \sum_{x \in \mathcal{G}_1} \mathbf{1}_A(S_x)$; then, for any $\theta \geq 0$ and $t \geq 0$,

$$(3.19) \quad \text{card } \mathcal{P}_t = \sum_{n=0}^{\infty} \sum_{x \in \mathcal{G}_n} \mathbf{1}_{[0,t]}(S_x) \xi^x(t - S_x, \infty) \leq e^{\theta t} \sum_{x \in \mathcal{P}} e^{-\theta S_x} \nu_x.$$

Set $V_\theta = \sum_{x \in \mathcal{P}} e^{-\theta S_x} \nu_x$ and recall that $\mathcal{P}_t = \{o\}$ for $t < 0$. By (3.19) then

$$\sum_{x \in \tilde{\mathcal{P}}_{t-s}} \text{card } \mathcal{P}_{t-S_x}^x \leq \sum_{x \in \tilde{\mathcal{P}}_{t-s}} [e^{\theta(t-S_x)} V_\theta^x \mathbf{1}_{[0,t]}(S_x) + \mathbf{1}_{(t,\infty)}(S_x)].$$

Since $x \in \tilde{\mathcal{P}}_{t-s} \Rightarrow S_x > t-s$, we then have that

$$(3.20) \quad \sum_{x \in \tilde{\mathcal{P}}_{t-s}} \text{card } \mathcal{P}_{t-S_x}^x \leq e^{\theta s} \sum_{x \in \tilde{\mathcal{P}}_{t-s}} V_\theta^x \mathbf{1}_{[0,t]}(S_x) + \text{card } \tilde{\mathcal{P}}_{t-s}.$$

Now assume $t \geq s$ and decompose $\tilde{\mathcal{P}}_{t-s}$ as in (3.19) to obtain

$$(3.21) \quad \begin{aligned} \sum_{x \in \tilde{\mathcal{P}}_{t-s}} V_\theta^x \mathbf{1}_{[0,t]}(S_x) &= \sum_{n=0}^{\infty} \sum_{x \in \tilde{\mathcal{G}}_n} \mathbf{1}_{[0,t-s]}(S_x) \sum_{y \in \tilde{\mathcal{G}}_1^x} \mathbf{1}_{(t-s,t]}(S_{xy}) V_\theta^{xy} \\ &\leq e^{\zeta(t-s)} \sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} V_\theta^{xy} \end{aligned}$$

and

$$(3.22) \quad \text{card } \tilde{\mathcal{P}}_{t-s} \leq e^{\zeta(t-s)} \sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \tilde{\nu}_x,$$

for any $\zeta \geq 0$. Combining (3.20), (3.21) and (3.22) we finally have that

$$(3.23) \quad \sup_{t \geq s} \left\{ e^{-\zeta t} \sum_{x \in \tilde{\mathcal{P}}_{t-s}} \text{card } \mathcal{P}_{t-S_x}^x \right\} \leq e^{(\theta-\zeta)s} \sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1)$$

for all $\theta, \zeta \geq 0$. We will now show that the right-hand side of (3.23) has finite expectation for the choice $\zeta = \tilde{D} + \delta$ and $\theta = D + \delta$. This, in conjunction with Claim 2, will prove Claim 3 in the case $p = 1$.

Let $\mathcal{F}_n = \sigma(\Phi_x : |x| < n)$, for $n \geq 1$, where here $|x|$ denotes the length of the finite sequence x . Write

$$\mathbb{E} \left[\sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1) \right] = \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_n} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1) \right]$$

and condition first on $\mathcal{F}_{n+1}$ to obtain

$$\mathbb{E} \left[\sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1) \right] = \mathbb{E}[V_\theta + 1] \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_n} e^{-\zeta S_x} \tilde{\nu}_x \right],$$

since the V_θ^{xy} are independent of $\mathcal{F}_{n+1}$ and all have the distribution of V_θ . Now condition on $\mathcal{F}_n$ to obtain

$$\begin{aligned} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1) \right] &= (\mathbb{E}[V_\theta] + 1) \mathbb{E}[\tilde{\nu}] \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_n} e^{-\zeta S_x} \right] \\ &= \frac{(\mathbb{E}[V_\theta] + 1) \mathbb{E}[\tilde{\nu}]}{[1 - \tilde{m}(\zeta)]_+}. \end{aligned}$$

Similarly, $\mathbb{E}[V_\theta] = \mathbb{E}[\nu] [1 - m(\theta)]_+^{-1}$. Since $\tilde{m}(\tilde{D} + \delta) < 1$ and $m(D + \delta) < 1$,

$$\mathbb{E} \left[\sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1) \right] < \infty$$

for $\zeta = \tilde{D} + \delta$ and $\theta = D + \delta$. This, in conjunction with (3.23) and Claim 2, completes the proof in the case $p = 1$.

If $p > 1$, then $\{\tilde{\mathcal{G}}_n : n \geq 0\}$ is not a Galton-Watson process anymore. However, $\{\mathcal{G}_n^{(p)} : n \geq 0\}$ is, where $\mathcal{G}_0^{(p)} = \{o\}$ and, for $n \geq 1$, $\mathcal{G}_n^{(p)}$ consists of those sequences in $\mathcal{G}_{np}$ for which, for all $1 \leq k \leq n$, there exists a $v \in \phi_{x_1 \dots x_{(k-1)p+1}} \circ \dots \circ \phi_{x_1 \dots x_{kp}}(J)$, with $\text{dist}(v, G^c) \leq \alpha$. Furthermore, $\tilde{\mathcal{G}}_{np} \subseteq \mathcal{G}_n^{(p)}$, for all n .

Inequality (3.23) is still valid and it only remains to show that the expectation of the right-hand side of (3.23) is finite. Conditioning again first on $\mathcal{F}_{n+1}$ and then on $\mathcal{F}_n$, we have successively

$$\begin{aligned} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{P}}} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} (V_\theta^{xy} + 1) \right] &= (\mathbb{E}[V_\theta] + 1) \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_n} e^{-\zeta S_x} \sum_{y \in \tilde{\mathcal{G}}_1^x} 1 \right] \\ &\leq (\mathbb{E}[V_\theta] + 1) \mathbb{E}[\nu] \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_n} e^{-\zeta S_x} \right]. \end{aligned}$$

Next observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_n} e^{-\zeta S_x} \right] &= \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_{np}} \sum_{k=0}^{p-1} \sum_{y \in \tilde{\mathcal{G}}_k^x} e^{-\zeta S_x} e^{-\zeta(S_{xy} - S_x)} \right] \\ &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{p-1} [m(\zeta)]^k \mathbb{E} \left[\sum_{x \in \mathcal{G}_n^{(p)}} e^{-\zeta S_x} \right] \\ &= \frac{[m(\zeta)]^p - 1}{m(\zeta) - 1} \sum_{n=0}^{\infty} \left(\mathbb{E} \left[\sum_{x \in \mathcal{G}_1^{(p)}} e^{-\zeta S_x} \right] \right)^n. \end{aligned}$$

Now take $\zeta = \tilde{D} + \delta$ and $\theta = D + \delta$ again, observe that

$$\mathbb{E} \left[\sum_{x \in \mathcal{G}_1^{(p)}} e^{-\zeta S_x} \right] = \mathbb{E} \left[\sum_{x \in \tilde{\mathcal{G}}_p} e^{-\zeta S_x} \right] < 1$$

for $\zeta = \tilde{D} + \delta$, by (3.18), and conclude the proof as in the case $p = 1$. $\square$

We now conclude the proof of Lemma 3.6. Recall (3.7).

$$R_t = \sum_{x \in \mathcal{G}_1} |K_x(e^{-t})| - |K(e^{-t})| \leq \sum_{x \in \mathcal{G}_1} \sum_{y \in \mathcal{G}_1 \setminus \{x\}} |K_x(e^{-t}) \cap K_y(e^{-t})|,$$

where recall $K_x = K \cap \phi_x(J)$. Let $Q_x(2\epsilon)$ denote the maximum cardinality of a 2ϵ -separated subset F_x of K_x , such that $\text{dist}(v, \phi_x(G^c)) \leq 2\epsilon$ for all $v \in F_x$. Then,

$$\bigcup_{y \in \mathcal{G}_1 \setminus \{x\}} K_x(e^{-t}) \cap K_y(e^{-t}) \subseteq \bigcup_{v \in F_x} B_{3\epsilon^{-t}}(v),$$

so that

$$R_t \leq \sum_{x \in \mathcal{G}_1} Q_x(2\epsilon^{-t})(3\epsilon^{-t})^d c_d.$$

Now, given that $x \in \mathcal{G}_1$, $Q_x(2\epsilon) = Q^x(2\epsilon e^{S_x})$, where for $\beta > 0$, $Q^x(\beta)$ is the maximum cardinality of a β -separated subset F^x of $K^x = \phi_x^{-1}(K_x)$, such that

$\text{dist}(v, G^c) \leq \beta$ for all $v \in F^x$. Furthermore, $Q^x(\beta)$ has the same distribution as $Q(\beta)$, for each $\beta > 0$. Hence

$$R_t \leq c_d 3^d e^{-dt} \sum_{x \in \mathcal{G}_1} Q^x(2e^{-t+S_x})$$

and therefore, for any $0 < \delta \leq (D - \tilde{D})/2$,

$$(3.24) \quad \sup_{t \geq 0} e^{(d-D+\delta)t} R_t \leq c_d 3^d 2^{-(\tilde{D}+\delta)} \sum_{x \in \mathcal{G}_1} e^{-(\tilde{D}+\delta)S_x} \sup_{0 < \epsilon \leq 2e^{S_x}} \epsilon^{\tilde{D}+\delta} Q^x(\epsilon).$$

We will show that the expectation of the right-hand side of (3.24) is finite for any such δ and this will prove Lemma 3.6 with $h(t) = e^{-\delta t} \mathbf{1}_{[0, \infty)}(t)$.

Since J is compact and $K^x \subseteq J$, for all $x \in \mathcal{G}_1$, there exists a finite deterministic constant C , such that $Q^x(\epsilon) \leq C$ for all $\epsilon > \alpha$ and all $x \in \mathcal{G}_1$. Thus

$$\sum_{x \in \mathcal{G}_1} e^{-(\tilde{D}+\delta)S_x} \sup_{0 < \epsilon \leq 2e^{S_x}} \epsilon^{\tilde{D}+\delta} Q^x(\epsilon) \leq \sum_{x \in \mathcal{G}_1} e^{-(\tilde{D}+\delta)S_x} \sup_{0 < \epsilon \leq \alpha} \epsilon^{\tilde{D}+\delta} Q^x(\epsilon) + \nu 2^{\tilde{D}+\delta} C,$$

and since $\mathbb{E}[\nu]$ is finite, it suffices to show that

$$\mathbb{E} \left[\sum_{x \in \mathcal{G}_1} e^{-(\tilde{D}+\delta)S_x} \sup_{0 < \epsilon \leq \alpha} \epsilon^{\tilde{D}+\delta} Q^x(\epsilon) \right] < \infty.$$

But, by conditioning on $\mathcal{F}_1$,

$$\begin{aligned} \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} e^{-(\tilde{D}+\delta)S_x} \sup_{0 < \epsilon \leq \alpha} \epsilon^{\tilde{D}+\delta} Q^x(\epsilon) \right] &= \mathbb{E} \left[\sup_{0 < \epsilon \leq \alpha} \epsilon^{\tilde{D}+\delta} Q(\epsilon) \right] \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} e^{-(\tilde{D}+\delta)S_x} \right] \\ &= \mathbb{E} \left[\sup_{0 < \epsilon \leq \alpha} \epsilon^{\tilde{D}+\delta} Q(\epsilon) \right] m(\tilde{D} + \delta) \end{aligned}$$

and this is finite by Claim 3. $\square$

3.4. Convergence in mean. Theorem 3.1 deals with the almost sure limiting behavior of the quantities $\epsilon^{-(d-D)} |K(\epsilon)|$ and $T^{-1} \int_0^T e^{(d-D)t} |K(e^{-t})| dt$. One can obtain results concerning the limiting behavior of the means of these quantities via the ordinary renewal theorem, as in the deterministic case.

Recall (3.7), (3.8) and (3.9). The function $w(t) = \mathbb{E}[W_t]$ satisfies the ordinary renewal equation

$$w = f + w * F,$$

where $*$ denotes convolution, F is the probability distribution function

$$(3.25) \quad F(t) = \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} e^{-DS_x} \mathbf{1}_{(-\infty, t]}(S_x) \right]$$

and $f(t) = e^{-Dt} \mathbb{E}[Z_t]$.

Now assume the hypotheses of Theorem 3.1 hold. By (3.11) we have that

$$\sum_{k=1}^{\infty} \sup_{s \in [k-1, k]} |f(s)| < \infty.$$

Furthermore, f is continuous except at a countable number of points. This is because the sample paths of Z have left- and right-hand limits and hence, by the

dominated convergence theorem (use (3.11) to dominate), f has left- and right-hand limits. It now follows that f is directly Riemann integrable over $[0, \infty)$. In particular f is also bounded.

By (3.16), w is bounded on finite intervals and therefore $w = f * U$, where $U = \sum_{n=0}^{\infty} F^{*n}$. By the ordinary renewal theorem then

$$\mathbb{E}[W_t] \longrightarrow \frac{1}{-m'(D)} \int_0^{\infty} e^{-Ds} \mathbb{E}[Z_s] ds,$$

as $t \longrightarrow \infty$, if $A \mapsto \mathbb{E}[\xi(A)]$ is non-lattice, while in the lattice case

$$\mathbb{E}[W_{s+n\lambda}] \longrightarrow \frac{\lambda e^{-Ds}}{-m'(D)} \sum_{k=0}^{\infty} e^{-Dk\lambda} \mathbb{E}[Z_{s+k\lambda}],$$

as $n \longrightarrow \infty$, for each $s \in [0, \lambda)$. As in the proof of Theorem 2.3 then, one always has that, as $T \longrightarrow \infty$,

$$\frac{1}{T} \int_0^T \mathbb{E}[W_t] dt \longrightarrow \frac{1}{-m'(D)} \int_0^{\infty} e^{-Ds} \mathbb{E}[Z_s] ds.$$

We summarize the above in the following theorem. Recall that $M_1 = \sum_{x \in \mathcal{G}_1} r_x^D$.

Theorem 3.7. (i) *Under the hypotheses of Theorem 3.1*

$$\frac{1}{T} \int_0^T e^{(d-D)t} \mathbb{E}[|K(e^{-t})|] dt \longrightarrow \frac{1}{-m'(D)} \int_0^{\infty} e^{-Ds} \mathbb{E}[Z_s] ds,$$

as $T \longrightarrow \infty$; if in addition $A \mapsto \mathbb{E}[\xi(A)]$ is non-lattice, then also, as $\epsilon \rightarrow 0$,

$$\epsilon^{-(d-D)} \mathbb{E}[|K(\epsilon)|] \longrightarrow \frac{1}{-m'(D)} \int_0^{\infty} e^{-Ds} \mathbb{E}[Z_s] ds.$$

(ii) *If in addition $\mathbb{E}[M_1 \log^+ M_1] < \infty$, then both, the convergence in Theorem 3.1(i) (in the non-lattice case) and the convergence in Theorem 3.1 (ii), hold also in L^1 .*

Proof. Assertion (i) has already been established above. Under the additional assumption $\mathbb{E}[M_1 \log^+ M_1] < \infty$, we also have, by Theorem 3.3, that $\mathbb{E}[M_{\infty}] = 1$ and so $\mathbb{E}[W_t] \longrightarrow \mathbb{E}[\lim W_t]$, in the non-lattice case, where $\lim W_t$ denotes the almost sure limit in (i) of Theorem 3.1. This, in the presence of the almost sure convergence of Theorem 3.1 (i) and non-negativity of W_t , implies the L^1 -convergence in the non-lattice case (see Billingsley [4], Corollary following Theorem 16.14).

Similarly, to establish the L^1 -convergence of $T^{-1} \int_0^T W_t dt$ to its a.s.-limit, observe that by (i) of the present theorem, Theorem 3.1 (ii) and the fact that $\mathbb{E}[M_{\infty}] = 1$, we have

$$\mathbb{E} \left[\frac{1}{T} \int_0^T W_t dt \right] = \frac{1}{T} \int_0^T \mathbb{E}[W_t] dt \longrightarrow \mathbb{E} \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T W_t dt \right].$$

Again this, in the presence of the corresponding a.s.-convergence and non-negativity, implies convergence in L^1 . $\square$

3.5. Proof of Theorem 3.2. (i) When $\mathbb{E}[M_1 \log^+ M_1] = \infty$, we have that $M_\infty = 0$ with probability equal to 1, by Theorem 3.3, so $L = 0$ with probability equal to 1, by the representation $L = M_\infty \times \text{constant}$ (see (3.12) and (3.15)).

(ii) Assume now $\mathbb{E}[M_1 \log^+ M_1] < \infty$. By our assumption (3.2), namely that $\mathbb{P}(K \cap G \neq \emptyset) > 0$, there exist a $\delta > 0$, and an event (i.e., a measurable set) A_δ , such that $\mathbb{P}(A_\delta) > 0$ and

$$\text{for } \omega \in A_\delta \quad \exists v(\omega) \in K(\omega) \text{ s.t. } B_\delta(v(\omega)) \subset G,$$

where, recall, $B_\delta(v)$ denotes the ball with center v and radius δ . Fix this δ for the rest of the proof.

Next fix $a > 0$, to be specified later in the proof. We claim that, on A_δ ,

$$(3.26) \quad |K(\delta e^{-t-a})| \geq c_d (\delta e^{-t-a})^d \sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x),$$

where, recall,

$$\mathcal{P}_t = \{x = x_1 \dots x_n \in \mathcal{P} : S_{x_1 \dots x_{n-1}} \leq t < S_{x_1 \dots x_n}\}.$$

This is because, for distinct $x = x_1 \dots x_n$ and $y = y_1 \dots y_m$ in $\mathcal{P}_t$, we must have $x_p \neq y_p$ for some $p \leq \min\{n, m\}$; then however,

$$\phi_{x_1} \circ \dots \circ \phi_{x_{n-1}}(G) \cap \phi_{y_1} \circ \dots \circ \phi_{y_{m-1}}(G) = \emptyset.$$

On A_δ , we may choose a point $v \in K$ such that $B_\delta(v) \subset G$, and then

$$B_{\delta e^{-S_x}}(v_x) \cap B_{\delta e^{-S_y}}(v_y) = \emptyset,$$

where $v_x = \phi_{x_1} \circ \dots \circ \phi_{x_{n-1}}(v)$ and $v_y = \phi_{y_1} \circ \dots \circ \phi_{y_{m-1}}(v)$. But if $x \in \mathcal{P}_t$, with $S_x \leq t + a$, then $\delta e^{-t-a} \leq \delta e^{-S_x}$, and so the balls $B_{\delta e^{-t-a}}(v_x)$, $x \in \mathcal{P}_t$ with $t < S_x \leq t + a$, are disjoint. This proves (3.26).

Now write

$$\sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x) = \sum_{x \in \mathcal{P}} Z_{t-S_x}^x,$$

with $Z_t = \xi(t, t+a]$, if $t \geq 0$, and $Z_t = 0$ otherwise, where recall $\xi(A) = \sum_{x \in \mathcal{G}_1} \mathbf{1}_A(S_x)$. We may apply Theorem 3.4, since obviously $\sup_{t \geq 0} Z_t \leq \nu$, and condition (3.4) holds with $h(t) = e^{-Dt} \mathbf{1}_{[0, \infty)}(t)$. We conclude that, in the non-lattice case, with probability equal to 1,

$$(3.27) \quad e^{-Dt} \sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x) \longrightarrow \frac{M_\infty}{-m'(D)} \int_0^\infty e^{-Du} \mathbb{E}[\xi(u, u+a)] du.$$

Write the limit above as $M_\infty C$, and notice that

$$\begin{aligned} C &= \frac{1}{-m'(D)} \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} \int_{\max\{0, S_x - a\}}^{S_x} e^{-Du} du \right] \\ &= \frac{1}{-Dm'(D)} \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} \{(e^{Da} - 1)e^{-DS_x} \mathbf{1}_{(a, \infty)}(S_x) + (1 - e^{-DS_x}) \mathbf{1}_{(0, a]}(S_x)\} \right], \end{aligned}$$

so

$$(3.28) \quad C > 0$$

for whatever choice of $a > 0$ we make; so fix some $a > 0$ for definiteness.

Now note that the convergence in (3.27) also holds in L^1 . To see this let $x(t)$ denote the expected value of $e^{-Dt} \sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x)$, for $t \geq 0$, and note that it satisfies the ordinary renewal equation $x = f + x * F$, where F is the distribution function defined in (3.25), and $f(t) = \mathbb{E}[\xi(t, t+a)]$, for $t \geq 0$. It follows, by the (ordinary) renewal theorem that

$$(3.29) \quad e^{-Dt} \mathbb{E} \left[\sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x) \right] \longrightarrow C.$$

It now follows by (3.27), (3.29) and non-negativity, that the convergence in (3.27) also holds in L^1 (see Billingsley [4], Corollary following Theorem 16.14) and therefore we also have

$$(3.30) \quad e^{-Dt} \mathbb{E} \left[\mathbf{1}_{A_\delta} \sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x) \right] \longrightarrow C \mathbb{E} [M_\infty \mathbf{1}_{A_\delta}].$$

Note that, since $A_\delta \subseteq \{K \neq \emptyset\}$ and $\mathbb{P}(M_\infty > 0 | K \neq \emptyset) = 1$, we have that $M_\infty > 0$ almost surely on A_δ . Thus

$$(3.31) \quad \mathbb{E} [M_\infty \mathbf{1}_{A_\delta}] > 0,$$

since we chose δ so that $\mathbb{P}(A_\delta) > 0$. Combining this with (3.26), (3.28) and (3.30) we then have that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon^{-(d-D)} \mathbb{E}[|K(\epsilon)|] &\geq \liminf_{t \rightarrow \infty} e^{(d-D)(t+a)} \delta^{-(d-D)} \mathbb{E}[\mathbf{1}_{A_\delta} |K(\delta e^{-t-a})|] \\ &\geq c_d \delta^D e^{-Da} \lim_{t \rightarrow \infty} e^{-tD} \mathbb{E} \left[\mathbf{1}_{A_\delta} \sum_{x \in \mathcal{P}_t} \mathbf{1}_{(t, t+a]}(S_x) \right] \\ &= c_d \delta^D e^{-Da} C \mathbb{E} [M_\infty \mathbf{1}_{A_\delta}] > 0. \end{aligned}$$

However, by Theorem 3.7 and (3.12), and since we are assuming $\mathbb{E}[M_1 \log^+ M_1] < \infty$, we have that

$$L = \lim_{\epsilon \rightarrow 0} \epsilon^{-(d-D)} |K(\epsilon)| = M_\infty \lim_{\epsilon \rightarrow 0} \epsilon^{-(d-D)} \mathbb{E}[|K(\epsilon)|]$$

and we conclude that $L > 0$ iff $M_\infty > 0$ in the non-lattice case. The result follows from Theorem 3.3.

For the lattice case choose $a = k\lambda$, where $\lambda > 0$ is the span and k is such that $\mathbb{P}(S_x = k\lambda \text{ some } x \in \mathcal{G}_1) > 0$, but $\mathbb{P}(S_x = j\lambda \text{ some } x \in \mathcal{G}_1) = 0$ for all $0 \leq j < k$. Theorem 3.4 implies that, for any fixed $s \in [0, \lambda)$, as $n \rightarrow \infty$,

$$(3.32) \quad e^{-Dn\lambda} \sum_{x \in \mathcal{P}_{s+n\lambda}} \mathbf{1}_{(s+n\lambda, s+n\lambda+a]}(S_x) \longrightarrow \frac{\lambda M_\infty}{-m'(D)} \sum_{i=0}^{\infty} e^{-D\lambda i} \mathbb{E}[\xi(s+i\lambda, s+i\lambda+a)],$$

with probability equal to 1. Write again the limit as $M_\infty C(s)$; by our choice of a

$$\begin{aligned} C(s) &= \frac{\lambda}{-m'(D)} \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} \sum_{\max\{0, S_x - a - s\} \leq n\lambda < S_x - s} e^{-Dn\lambda} \right] \\ &\geq \frac{\lambda}{-m'(D)} \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} \mathbf{1}_{\{k\lambda\}}(S_x) \sum_{\max\{0, S_x - a - s\} \leq n\lambda < S_x - s} e^{-Dn\lambda} \right] \\ &= \frac{\lambda}{-m'(D)} \frac{1 - e^{-kD\lambda}}{1 - e^{-D\lambda}} \mathbb{E}[\xi(\{k\lambda\})], \end{aligned}$$

for $s \in [0, \lambda)$. Let C denote the last expression above; then

$$(3.33) \quad C(s) \geq C > 0,$$

for all $s \in [0, \lambda)$. As in the non-lattice case, the convergence in (3.32) also holds in L^1 and therefore, as $n \rightarrow \infty$,

$$(3.34) \quad e^{-Dn\lambda} \mathbb{E} \left[\mathbf{1}_{A_\delta} \sum_{x \in \mathcal{P}_{s+n\lambda}} \mathbf{1}_{(s+n\lambda, s+n\lambda+a]}(S_x) \right] \rightarrow C(s) \mathbb{E}[M_\infty \mathbf{1}_{A_\delta}],$$

for each $s \in [0, \lambda)$. Now

$$\begin{aligned} &\inf_{s \in [0, \lambda)} \lim_{n \rightarrow \infty} e^{(s+n\lambda)(d-D)} \mathbb{E}[|K(e^{-s-n\lambda})|] \\ &= \inf_{s \in [0, \lambda)} \lim_{n \rightarrow \infty} \delta^{-(d-D)} e^{(s+n\lambda+a)(d-D)} \mathbb{E}[|K(\delta e^{-s-n\lambda-a})|] \\ &\geq \inf_{s \in [0, \lambda)} \lim_{n \rightarrow \infty} \delta^{-(d-D)} e^{(s+n\lambda+a)(d-D)} \mathbb{E}[\mathbf{1}_{A_\delta} |K(\delta e^{-s-n\lambda-a})|] \end{aligned}$$

and we then have successively by (3.26), (3.34), (3.33) and (3.31)

$$\begin{aligned} &\inf_{s \in [0, \lambda)} \lim_{n \rightarrow \infty} e^{(s+n\lambda)(d-D)} \mathbb{E}[|K(e^{-s-n\lambda})|] \\ &\geq c_d \delta^D e^{-aD} \inf_{s \in [0, \lambda)} \lim_{n \rightarrow \infty} e^{-(s+n\lambda)D} \mathbb{E} \left[\mathbf{1}_{A_\delta} \sum_{x \in \mathcal{P}_{s+n\lambda}} \mathbf{1}_{(s+n\lambda, s+n\lambda+a]}(S_x) \right] \\ &= c_d \delta^D e^{-aD} \mathbb{E}[M_\infty \mathbf{1}_{A_\delta}] \inf_{s \in [0, \lambda)} e^{-Ds} C(s) \\ &\geq c_d \delta^D e^{-(a+\lambda)D} \mathbb{E}[M_\infty \mathbf{1}_{A_\delta}] C > 0. \end{aligned}$$

But by Theorem 3.7 again, and (3.15),

$$\begin{aligned} L &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{(d-D)t} |K(e^{-t})| dt \\ &= M_\infty \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{(d-D)t} \mathbb{E}[|K(e^{-t})|] dt \\ &= M_\infty \frac{1}{\lambda} \int_0^\lambda \lim_{n \rightarrow \infty} e^{(s+n\lambda)(d-D)} \mathbb{E}[|K(e^{-s-n\lambda})|] ds \\ &\geq M_\infty \inf_{s \in [0, \lambda)} \lim_{n \rightarrow \infty} e^{(s+n\lambda)(d-D)} \mathbb{E}[|K(e^{-s-n\lambda})|] \end{aligned}$$

and we conclude that $L > 0$ iff $M_\infty > 0$ for the lattice case as well. By Theorem 3.3 this concludes the proof of the theorem. $\square$

4. EXAMPLES

4.1. **Cantor set.** As already mentioned after Theorem 2.3, one cannot apply part (i) of this theorem to the usual ternary Cantor set. For this set $r_1 = r_2 = 1/3$ and so $\{-\log r_1, -\log r_2\} \subset \lambda\mathbb{Z}$, with $\lambda = \log 3$; here $D = \log_3 2$. Using (2.9), one can actually compute the limits

$$L(s) = \lim_{n \rightarrow \infty} e^{(1-D)(s+n\lambda)} |K(e^{-s-n\lambda})|,$$

along sequences $s + n\lambda$, $s \in [0, \log 3)$; one finds

$$L(s) = \begin{cases} e^{(1-D)s} \left(\frac{3}{2} + e^{-s}\right) & \text{for } 0 \leq s < \log 2, \\ e^{(1-D)s} (1 + 2e^{-s}) & \text{for } \log 2 \leq s < \log 3. \end{cases}$$

Thus $e^{-(d-D)} |K(\epsilon)|$ does *not* have a limit in this case.

(For the usual ternary Cantor set this fact is well known and can be obtained by direct computation, without the aid of Theorem 2.3; see for example [15]; see also the Acknowledgments at the end of section 1.)

4.2. **Fractal percolation (Mandelbrot's canonical curdling).** Let $d \geq 1$ and fix an integer $N \geq 2$ and $p \in (0, 1)$, with $N^d p > 1$. We consider the random set K obtained as follows. Divide the unit (hyper)cube $[0, 1]^d$ into the N^d congruent subcubes $\prod_{k=1}^d [i_k N^{-1}, (i_k + 1)N^{-1}]$, $0 \leq i_k < N$, $1 \leq k \leq d$, and keep each of them with probability p (discard it with probability $1 - p$), independently of all other subcubes. Next divide each of the subcubes kept in the first step into the N^d congruent subcubes of side-length N^{-2} it contains and keep each of these with probability p , independently of all else. The set K is then obtained by continuing this process ad-infinitum.

The set K is a stochastically self-similar set as described in subsection 3.1, with $J = [0, 1]^d$, $G = (0, 1)^d$ and $\Phi = \{\phi_1, \dots, \phi_\nu\}$ such that each ϕ_i is one of the similarities

$$\psi_{i_1 \dots i_d}(v) = \frac{1}{N} [v + (i_1, \dots, i_d)^T], \quad v \in \mathbb{R}^d, \quad 0 \leq i_k < N, \quad 1 \leq k \leq d,$$

where T denotes transpose. In this case $\mathbb{E}[\nu] = N^d p$ and this explains our assumption $N^d p > 1$. The dimension of K , almost surely on the event $\{K \neq \emptyset\}$, is the solution in D of the equation

$$1 = \mathbb{E} \left[\sum_{x \in \mathcal{G}_1} e^{-DS_x} \right] = N^{-D} \mathbb{E}[\nu];$$

so $D = d + \log_N p$. Furthermore, the measure $A \mapsto \mathbb{E}[\xi(A)]$ is lattice, with span $\lambda = \log N$, and Theorem 3.1 (ii) applies, as clearly

$$(4.1) \quad \mathbb{P}(K \cap G \neq \emptyset) > 0.$$

Note that the assumption that we keep subcubes *independently* of one-another is irrelevant to this discussion and we could have any other distribution for Φ , provided (4.1) is satisfied (independence between successive steps is still required however); the value of $\mathbb{E}[\nu]$, and hence also of D , would then be different of course.

Note also that, because ν is bounded here, Theorem 3.7 (ii) also applies.

4.3. Level-sets of stable processes. Let $\{X_t : t \geq 0\}$ be a real-valued stable process of index $\alpha \in (0, 2]$, started at 0. This is a process with stationary and independent increments (Levy process), which has the following scaling property: for every $s > 0$, the process $\{s^{-1/\alpha}X_{st} : t \geq 0\}$ has the same law (distribution) as X itself. For $\alpha = 2$, the process X is a constant multiple of standard Brownian motion.

Let $K = \{s \geq 0 : X_s = 0\}$, the zero-set of X , and $K_t = K \cap [0, t]$, $t \geq 0$, the zero-set of X up to time t . It is well known that, when $1 < \alpha \leq 2$, K and each K_t are nonempty closed perfect sets (because 0 is regular for itself, by the scaling property), of Hausdorff dimension $D = 1 - 1/\alpha$ ([5], [24]). When $0 < \alpha \leq 1$ on the other hand, $K = \{0\}$, as singletons are polar in this case (see [2]). We therefore restrict attention to the case $1 < \alpha \leq 2$ henceforth. We will explain how to apply Theorems 3.1 and 3.7 to obtain the following statement: *for each fixed $t \geq 0$, the limit*

$$(4.2) \quad L_t = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} |K_t(\epsilon)|$$

exists a.s. and this convergence also holds in L^1 .

Fix $t > 0$ and let $\{Y_s : 0 \leq s \leq t\}$ be the corresponding stable bridge; this, intuitively, is the process $X|_{[0,t]}$ conditioned to be 0 at time t . Denote by K_Y the zero set of Y : $K_Y = \{s \in [0, t] : Y_s = 0\}$. The set K_Y is then a stochastically self-similar set in the sense of subsection 3.1, with $J = [0, t]$ and $\Phi = \{\phi_1, \phi_2\}$, where ϕ_1 and ϕ_2 are the similarities of $\mathbb{R}$ given by

$$\phi_1(v) = \frac{\tau_1}{t}v, \quad \phi_2(v) = \frac{\tau_2}{t}v + t - \tau_2,$$

and $\tau_1 = \sup\{s \in [0, t/2] : Y_s = 0\}$ and $t - \tau_2 = \inf\{s \in [t/2, t] : Y_s = 0\}$. Clearly, condition (3.2) is satisfied, as, for example, $\tau_1 \in K \cap G$. Also, the measure $A \mapsto \mathbb{E}[\xi(A)]$ is non-lattice, as τ_1 and τ_2 have (actually identical) absolutely continuous distributions. Finally, K_Y also has dimension $1 - 1/\alpha$. One way to see this is to observe that

$$(4.3) \quad K_Y \stackrel{\mathcal{L}}{=} \frac{t}{\tau} K_t,$$

where $\stackrel{\mathcal{L}}{=}$ means ‘equal in law’ and $\tau = \sup\{s \in [0, t] : X_s = 0\}$. This is because the processes Y and $\{(\tau/t)^{-1/\alpha}X_{s\tau/t} : 0 \leq s \leq t\}$ have the same laws (i.e., the latter is a bridge), by the scaling property (see [2], Theorem VIII.12). It now follows from Theorem 3.1 that, with probability equal to one, the limit

$$(4.4) \quad L_Y = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} |K_Y(\epsilon)|$$

exists and this convergence also holds in L^1 , by Theorem 3.7. Statement (4.2) then follows from this convergence, using the representation (4.3) and observing that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} |K_t(\epsilon)| &= \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} \frac{\tau}{t} \left| \left(\frac{t}{\tau} K_t \right) \left(\frac{\epsilon t}{\tau} \right) \right| \\ &= \left(\frac{\tau}{t} \right)^{1-1/\alpha} \lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon t}{\tau} \right)^{-1/\alpha} \left| \left(\frac{t}{\tau} K_t \right) \left(\frac{\epsilon t}{\tau} \right) \right|. \end{aligned}$$

As a matter of fact, taking $Y_s = (\tau/t)^{-1/\alpha}X_{s\tau/t}$, $0 \leq s \leq t$, as we may, we obtain that $L_t = (\tau/t)^{1-1/\alpha}L_Y$; furthermore τ is independent of L_Y ([2], Theorem VIII.12).

Clearly, one can then deal similarly with the case where the starting point X_0 is an arbitrary point $x \in \mathbb{R}$, by considering the first time the process hits 0 and using the strong Markov property, and also with arbitrary level-sets $K^y = \{s \geq 0 : X_s = y\}$ and $K_t^y = K^y \cap [0, t]$, $t \geq 0$, $y \in \mathbb{R}$, by considering the first time the process hits y .

Remark. Statement (4.2) is not new. In the case of Brownian motion, which is stable with $\alpha = 2$, it was already known to P. Levy, as can be deduced from section 2.2 of Ito and McKean ([11]); in particular, it follows directly from 5) and 6) on page 43, reference [11]. The general case $1 < \alpha \leq 2$, follows similarly from the results in section 7 of Fristedt and Taylor ([9]). Furthermore, the results in [9] imply that the limit L_t is a constant multiple of the local time at 0.

One can actually deduce a stronger statement than the a.s. convergence in (4.2) from Theorem 3.1.

Theorem 4.1. *Let $\{X_t : t \geq 0\}$ be a real-valued stable process, of index $\alpha \in (1, 2]$. Then, there exists a set of probability one, on which $\epsilon^{-1/\alpha}|K_t(\epsilon)|$ converges, as $\epsilon \rightarrow 0$, simultaneously for all $t \geq 0$. Furthermore, the limit L_t , $t \geq 0$, is continuous in t .*

Remark. It is plain that, given its existence, the limit $\{L_t : t \geq 0\}$ is adapted and constitutes an additive functional. As such, and being continuous, it must be a constant multiple of the local time of X at 0, by Proposition IV.5 of reference [2]. Reversing the point of view, one can define the local time process by $L_t = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha}|K_t(\epsilon)|$, $t \geq 0$, without a priori knowledge of its existence.

Proof of Theorem 4.1. Fix $t > 0$ and consider the bridge $Y_s = (\tau/t)^{-1/\alpha} X_{s\tau/t}$, $0 \leq s \leq t$, its zero-set K_Y and set $K_{Y,s} = K_Y \cap [0, s]$. In the notation of subsection 3.1, we have that $\mathcal{G}_n = \{1, 2\}^n$, for all $n \geq 0$, and $\mathcal{P} = \bigcup_{n=0}^{\infty} \{1, 2\}^n$. Furthermore, since $\mathcal{P}$ is countable, the limit

$$(4.5) \quad L_Y^x = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} |K_Y^x(\epsilon)|$$

exists for all $x \in \mathcal{P}$ simultaneously on a set Ω'_t of probability one, and one has the following equality, for each $x \in \mathcal{P}$ and all $n \geq 0$:

$$(4.6) \quad e^{-DS_x} L_Y^x = \sum_{y \in \mathcal{G}_n^x} e^{-DS_{xy}} L_Y^{xy},$$

with $D = 1 - 1/\alpha$. Define random functions $\{L_{Y,s}^{(n)} : 0 \leq s \leq t\}$, by letting $L_{Y,\cdot}^{(n)}$ be the distribution function of the measure that spreads mass $e^{-DS_x} L_Y^x$ uniformly on the interval $\phi_x([0, t])$, for $x \in \mathcal{G}_n$. By (4.6) then $L_{Y,t}^{(n)} = L_Y$, for all $n \geq 0$.

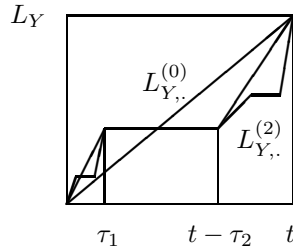
It is plain that (see Figure 4)

$$(4.7) \quad \left\| L_{Y,\cdot}^{(n+1)} - L_{Y,\cdot}^{(n)} \right\|_{\infty} \leq \max_{x \in \mathcal{G}_n} e^{-DS_x} L_Y^x,$$

for all $n \geq 0$. We next note the following lemma, whose proof we postpone to the end of this section.

Lemma 4.2.

$$\sum_{n=0}^{\infty} \mathbb{E} \left[\max_{x \in \mathcal{G}_n} e^{-DS_x} L_Y^x \right] < \infty.$$

FIGURE 4. The functions $L_{Y,\cdot}^{(0)}$, $L_{Y,\cdot}^{(1)}$ and $L_{Y,\cdot}^{(2)}$.

It now follows from the lemma that on a set $\Omega_t \subseteq \Omega'_t$, of probability one, the functions $L_{Y,\cdot}^{(n)}$ converge uniformly to a limit $L_{Y,\cdot}$, which must be continuous. Because $|K_{Y,r}(\epsilon)| \leq |K_{Y,s}(\epsilon)|$, for $0 \leq r < s \leq t$, because the limit $L_{Y,\cdot}$ is continuous and because, by (4.5),

$$(4.8) \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} |K_{Y,s}(\epsilon)| = L_{Y,s}$$

for those $s \in [0, t]$ which are endpoints of the intervals $\phi_x([0, t])$, $x \in \mathcal{P}$, and because such s are dense in K_Y , it follows that the convergence (4.8) holds simultaneously for all $s \in K_Y$. Finally, because both sides of (4.8) are constant functions of s on each of the intervals in $[0, t] \setminus K_Y$, (4.8) holds for all $s \in [0, t]$ simultaneously on Ω_t .

Using the representation (4.3) (which now holds pointwise and not just in law, by our choice of Y), we then have that, on Ω_t ,

$$L_s := \left(\frac{\tau}{t}\right)^{1-1/\alpha} L_{Y,s\tau/t} = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/\alpha} |K_s(\epsilon)|$$

for all $0 \leq s \leq \tau$; defining $L_s = L_\tau$, for $\tau < s \leq t$, one has that

$$(4.9) \quad \epsilon^{-1/\alpha} |K_s(\epsilon)| \longrightarrow L_s$$

for all $s \in [0, t]$ on Ω_t . Finally, the set $\Omega_\infty := \bigcap_{n \geq 1} \Omega_n$ has probability one and on it (4.9) holds for all $t \geq 0$ and the function $t \mapsto L_t$ is continuous. $\square$

Proof of Lemma 4.2. First note that $e^{-DS_x} \leq 2^{-n(1-1/\alpha)}$ for $x \in \mathcal{G}_n$. Hence

$$(4.10) \quad \mathbb{E} \left[\max_{x \in \mathcal{G}_n} e^{-DS_x} L_Y^x \right] \leq 2^{-n(1-1/\alpha)} E_{2^n},$$

where E_n denotes the expectation of the maximum of n independent copies of L_Y . We will show that, for any $\epsilon > 0$, there exists a constant $0 < C_\epsilon < \infty$, such that $E_n \leq C_\epsilon n^\epsilon$, $n \geq 0$, and this, in conjunction with (4.10), will prove the lemma.

Recall the martingale $M_n = \sum_{x \in \mathcal{G}_n} e^{-DS_x}$ and note that, since $M_1 \leq 2$, its limit M_∞ has finite moments of all orders, by Theorem 2.1 of reference [20]. Recall also, from (3.12), that $L_Y = M_\infty \times \text{const.}$; therefore L_Y has finite moments of all orders. It follows from Markov's inequality that, for any $\beta > 0$ and $u > 0$,

$$\mathbb{P}(L_Y > u) \leq u^{-\beta} \mathbb{E}[L_Y^\beta].$$

Fix $\beta > 1$ and let $c = \mathbb{E}[L_Y^\beta]^{1/\beta}$. Since $E_n = \int_0^\infty \{1 - [1 - \mathbb{P}(L_Y > u)]^n\} du$,

$$(4.11) \quad E_n \leq c + \int_c^\infty [1 - (1 - c^\beta u^{-\beta})^n] du = c \left[1 + \beta^{-1} \int_0^1 \frac{1 - u^n}{(1 - u)^{1+1/\beta}} du \right].$$

Using the beta-function,

$$\int_0^1 \frac{1 - u^n}{(1 - u)^{1+1/\beta}} du = \sum_{i=1}^n \int_0^1 u^{i-1} (1 - u)^{-1/\beta} du = \sum_{i=1}^n \frac{\beta}{i\beta - 1} \prod_{j=1}^{i-1} \frac{j\beta}{j\beta - 1},$$

where, by convention, $\prod_{i=1}^0 = 1$. Now, use the well known relation between sums and products, to obtain

$$\int_0^1 \frac{1 - u^n}{(1 - u)^{1+1/\beta}} du \leq C \sum_{i=2}^n \frac{\beta}{i\beta - 1} [(i-1)\beta - 1]^{1/\beta} + \frac{\beta}{\beta - 1} \leq C' n^{1/\beta},$$

for some finite constant C' , which, together with (4.11), prove the lemma. $\square$

5. CONCLUDING REMARKS

1. As should be clear from Example 4.1 (Cantor set), formulae (2.8) and (2.9) can be used for explicit computations, at least in low dimensions ($d = 1, 2$ say).
2. We conjecture that the non-lattice condition of Theorem 2.3 (i) is also necessary for $\epsilon^{-(d-D)}|K(\epsilon)|$ to converge to a limit as $\epsilon \rightarrow 0$, at least in non-trivial cases. To be more precise, let I denote the convex hull of K .

Conjecture. If

$$(5.1) \quad I \setminus \bigcup_{i=1}^N \phi_i(I) \neq \emptyset,$$

then $\epsilon^{-(d-D)}|K(\epsilon)|$ converges to a limit, as $\epsilon \rightarrow 0$, iff

$$\{\log r_1, \dots, \log r_N\} \not\subseteq \lambda \mathbb{Z} \quad \text{for any } \lambda > 0.$$

Notice that (5.1) is *not* satisfied for the set of similarities $\phi_1(x) = x/3$, $\phi_2(x) = (x+1)/3$, $\phi_3(x) = (x+2)/3$. The invariant set K for these similarities is $K = [0, 1]$, $D = 1$, and obviously $\epsilon^{-(1-D)}|K(\epsilon)| \rightarrow 1$, as $\epsilon \rightarrow 0$. For specific cases of 'lattice self-similar sets', one can verify the conjecture as in Example 4.1, but it would be nice to have a general argument. (*Note added in proof:* See also the recent article [26] by van Frankenhuysen and Lapidus; Remark 5.2 there verifies the conjecture for a certain class of self-similar subsets of $\mathbb{R}$.)

3. The methods of this paper rely heavily on the self-similarity of K , i.e., the fact that each of the maps ϕ_i is a similarity transformation. One can consider the same problem for fractals with some *approximate* self-similarity, like limit sets of Fuchsian groups and Schottky groups, or hyperbolic Julia sets of rational maps. The problem is considerably harder for such sets, but similar methods apply.

APPENDIX

In subsection 3.1 we considered random elements $\Phi = \{\phi_1, \dots, \phi_\nu\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here we consider matters of measurability.

Recall that $\mathcal{S}$ stands for the set of all contracting similarities of $\mathbb{R}^d$ and J is a fixed compact set, with $G = \text{int}(J)$ and $\overline{G} = J$. Set $\mathcal{S}_J := \{s \in \mathcal{S} : s(J) \subseteq J\}$ and, for $n > 1$

$$\mathcal{S}_J^{(n)} := \{\{s_1, \dots, s_n\} : s_i(J) \subseteq J \text{ for all } i \text{ and } s_i(G) \cap s_j(G) = \emptyset \text{ for } i \neq j\}.$$

Then Φ takes values in $\mathcal{S}_J^* := \bigcup_{n=0}^{\infty} \mathcal{S}_J^{(n)}$, where $\mathcal{S}_J^{(1)} = \mathcal{S}_J$ and $\mathcal{S}_J^{(0)} = \{\mathbf{e}\}$.

Each element of $\mathcal{S}$ is of the form $s(v) = rAv + b$, with $r \in (0, 1)$, $A \in O(\mathbb{R}^d)$ and $b \in \mathbb{R}^d$. We may therefore endow $\mathcal{S}$ with a metric:

$$\text{dist}(s, s') := |r - r'| + \|A - A'\| + |b - b'|.$$

The resulting metric space is separable and complete and $\mathcal{S}_J$ is closed in it. We may also endow each $\mathcal{S}^{(n)}$, the set of all subsets of $\mathcal{S}$ with n elements, with a metric, namely

$$\text{dist}(S, S') = \min_{\pi} \max_{s \in S} \text{dist}(s, \pi(s)),$$

where the minimum is taken over all bijections $\pi : S \rightarrow S'$. Then, for each $n \geq 1$, $\mathcal{S}_J^{(n)}$ is closed in $\mathcal{S}^{(n)}$ and hence in $\mathcal{B}^{(n)}$, the Borel σ -algebra of $\mathcal{S}^{(n)}$. Therefore $\mathcal{B}_J^{(n)} := \{\mathcal{S}_J^{(n)} \cap B : B \in \mathcal{B}^{(n)}\}$ is a σ -algebra and we take

$$\mathcal{B}_J^* := \{B \in \mathcal{S}^* : B \cap \mathcal{S}_J^{(n)} \in \mathcal{B}_J^{(n)} \text{ for all } n \geq 0\}$$

as our σ -algebra on $\mathcal{S}_J^*$, where $\mathcal{B}_J^{(0)} := \{\emptyset, \mathcal{S}_J^{(0)}\}$ is the trivial σ -algebra. We then require of Φ that it be a map $\Phi : \Omega \rightarrow \mathcal{S}_J^*$, measurable with respect to $\mathcal{F}$ and $\mathcal{B}_J^*$.

ADDED AFTER POSTING

In the Conjecture in section 5, [26] should read [25].

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